Asymptotic matricial models and QWEP property for q-Araki-Woods von Neumann algebras

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Abstract. Using Speicher central limit Theorem we provide Hiai's q-Araki-Woods von Neumann algebras with nice asymptotic matricial models. Then, we use this model and an elaborated ultraproduct procedure, to show that all q-Araki-Woods von Neumann algebras are QWEP.

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1 Introduction

Recall that a C^*-algebra has the weak expectation property (in short WEP) if the canonical inclusion from A into A^{**} factorizes completely contractively through some $B(H)$ (H Hilbert). A C^*-algebra is QWEP if it is a quotient by a closed ideal of an algebra with the WEP. The notion of QWEP was introduced by Kirchberg in [Kir]. Since then, it became an important notion in the theory of C^*-algebras. Very recently, Pisier and Shlyakhtenko [PS] proved that Shlyakhtenko's free quasi-free factors are QWEP. This result plays an important role in their work on the operator space Grothendieck Theorem, as well as in the subsequent related works [P1] and [Xu]. On the other hand, in his paper [J] on the embedding of Pisier's operator Hilbertian space OH and the projection constant of OH_n, Junge used QWEP in a crucial way.

Hiai [Hi] introduced the so-called q-Araki-Woods algebras. Let $-1 < q < 1$, and let $H_{\mathbb{R}}$ be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ an orthogonal group on $H_{\mathbb{R}}$. Let $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ denote the associated q-Araki-Woods algebra. These algebras are generalizations of both Shlyakhtenko's free quasi-free factors (for $q = 0$), and Bożejko and Speicher's q-Gaussian algebras (for $(U_t)_{t\in\mathbb{R}}$ trivial). In this paper we prove that $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ is QWEP. This is an extension of Pisier-Shlyakthenko's result for the free quasi-free factor (with $(U_t)_{t\in\mathbb{R}}$ almost periodic), already quoted above.

In the first two sections below we recall some general background on q-Araki-Woods algebras and we give a proof of our main result in the particular case of Bożejko and Speicher's q-Gaussian algebras $\Gamma_q(H_{\mathbb{R}})$. The proof relies on an asymptotic random matrix model for standard q-Gaussians. The existence of such a model goes back to Speicher's central limit Theorem for mixed commuting/anti-commuting non-commutative random variables (see [Sp]). Alternatively, one can also use the Gaussian random matrix model given by Śniady in [Sn]. Notice that the matrices arising from Speicher's central limit Theorem may not be uniformly bounded in norm. Therefore, we have to cut them off in order to define a homomorphism from a dense subalgebra of $\Gamma_q(H_{\mathbb{R}})$ into an ultraproduct of matricial algebras. In this tracial framework it can be shown quite easily that this homomorphism extends to an isometric $*$-homomorphism of von Neumann algebras, simply because it is trace preserving. Thus $\Gamma_q(H_{\mathbb{R}})$ can be seen as a (necessarily completely complemented) subalgebra of an ultraproduct of matricial algebras. This solves the problem in the tracial case.

Moreover, in this (relatively) simple situation, we are able to extend the result to the C^*-algebra generated by all q-Gaussians, $C_q^*(H_{\mathbb{R}})$. Indeed, using the ultracontractivity of the q-Ornstein Uhlenbeck semi-group (see [B]) we establish that $C_q^*(H_{\mathbb{R}})$ is "weakly ucp complemented" in $\Gamma_q(H_{\mathbb{R}})$. This last fact, combined with the QWEP of $\Gamma_q(H_{\mathbb{R}})$, implies that $C_q^*(H_{\mathbb{R}})$ is also QWEP.

In the remaining of the paper we adapt the proof of section 3 to the more general type-*III* q-Araki-Woods algebras. In section 4 we start by recalling Raynaud's construction of the von Neumann algebra's ultraproduct when algebras are equipped with non-tracial states (see [Ray]). Then, we give some general conditions in order to define an embedding into such an ultraproduct, whose image is of a state preserving conditional expectation.

In section 5 we define a twisted Baby Fock model, to which we apply Speicher's central limit Theorem. This provides us with an asymptotic random matrix model for (finite dimensional) q-Araki Woods algebras, generalizing the asymptotic model already introduced by Speicher and used by Biane in [Bi]. Using this asymptotic model, we then define an algebraic

$*$-homomorphism from a dense subalgebra of $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ into a von Neumann ultraproduct of finite dimensional C^*-algebras. Notice that the cut off argument requires some extra work (compare the proofs of Lemma 3.1 and Lemma 5.7), for instance we need to use our knowledge of the modular theory at the Baby Fock level to conclude. We then apply the general results of section 4 (Theorem 4.3) to extend this algebraic $*$-homomorphism into a $*$-isomorphism from $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ to the von Neumann algebra's ultraproduct, whose image is completely complemented. This allows us to show that $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ is QWEP for $H_\mathbb{R}$ finite dimensional (see Theorem 5.8). It implies, by inductive limit, that $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ is QWEP when $(U_t)_{t\in\mathbb{R}}$ is almost periodic (see Corollary 5.9).

In the last section, we consider a general algebra $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$. We use a discretization procedure on the unitary group $(U_t)_{t\in\mathbb{R}}$ in order to approach $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ by almost periodic q-Araki-Woods algebras. We then apply the general results of section 4 and, we recover the general algebra as a complemented subalgebra of the ultraproduct of the discretized ones (see Theorem 6.3). From this last fact follows the QWEP of $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$. However we were unable to establish the corresponding result for the C^*-algebra $C_q^*(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$. Indeed, if $(U_t)_{t\in\mathbb{R}}$ is not trivial then the ultracontractivity of the q-Ornstein-Uhlenbeck semi-group never holds in any right-neighborhood of zero (see [Hi]).

We highlight that the modular theory on the twisted Baby Fock algebras, on their ultraproduct, and on the q-Araki Woods algebras, are crucial tools in order to overcome the difficulties arising in the non-tracial case.

After the completion of this work, Marius Junge informed us that he had obtained our main result using his proof of the non-commutative L^1-Khintchine inequalities for q-Araki-Woods algebras. Junge's approach is slightly different but its main steps are the same as ours: the proof uses in a crucial way Speicher's central limit Theorem, an ultraproduct argument and modular theory.

2 Preliminaries

2.1 q-Araki-Woods algebras

We mainly follow the notations used in [Sh], [Hi] and [Nou]. Let $H_\mathbb{R}$ be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous group of orthogonal transformations on $H_\mathbb{R}$. We denote by $H_\mathbb{C}$ the complexification of $H_\mathbb{R}$ and still by $(U_t)_{t\in\mathbb{R}}$ its extension to a group of unitaries on $H_\mathbb{C}$. Let A be the (unbounded) non degenerate positive infinitesimal generator of $(U_t)_{t\in\mathbb{R}}$.

$$U_t = A^{it} \qquad \text{for all } t\in\mathbb{R}$$

A new scalar product $\langle\ .\ ,\ .\ \rangle_U$ is defined on $H_\mathbb{C}$ by the following relation:

$$\langle \xi,\eta\rangle_U = \langle 2A(1+A)^{-1}\xi,\eta\rangle$$

We denote by H the completion of $H_\mathbb{C}$ with respect to this new scalar product. For $q\in(-1,1)$ we consider the q-Fock space associated with H and given by:

$$\mathcal{F}_q(H) = \mathbb{C}\Omega \bigoplus_{n\geqslant 1} H^{\otimes n}$$

where $H^{\otimes n}$ is equipped with Bożejko and Speicher's q-scalar product (see [BS1]). The usual creation and annihilation operators on $\mathcal{F}_q(H)$ are denoted respectively by a^* and a (see [BS1]). For $f \in H_{\mathbb{R}}$, $G(f)$, the q-Gaussian operator associated to f, is by definition:

$$G(f) = a^*(f) + a(f) \in B\left(\mathcal{F}_q(H)\right)$$

The von Neumann algebra that they generate in $B\left(\mathcal{F}_q(H)\right)$ is the so-called q-Araki-Woods algebra: $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$. The q-Araki-Woods algebra is equipped with a faithful normal state φ which is the expectation on the vacuum vector Ω. We denote by W the Wick product ; it is the inverse of the mapping:

$$\begin{aligned}\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}}) &\longrightarrow \Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})\Omega \\ X &\mapsto X\Omega\end{aligned}$$

Recall that $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}}) \subset B\left(\mathcal{F}_q(H)\right)$ is the GNS representation of (Γ, φ). The modular theory relative to the state φ was computed in the papers [Hi] and [Sh]. We now briefly recall their results. As usual we denote by S the closure of the operator:

$$S(x\Omega) = x^*\Omega \quad \text{for all } x \in \Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$$

Let $S = J\Delta^{\frac{1}{2}}$ be its polar decomposition. J and Δ are respectively the modular conjugation and the modular operator relative to φ. The following explicit formulas hold :

$$S(h_1 \otimes \cdots \otimes h_n) = h_n \otimes \cdots \otimes h_1 \quad \text{for all } h_1, \ldots, h_n \in H_{\mathbb{R}}$$

Δ is the closure of the operator $\overset{\infty}{\underset{n=0}{\oplus}} (A^{-1})^{\otimes n}$ and

$$J(h_1 \otimes \cdots \otimes h_n) = A^{-\frac{1}{2}}h_n \otimes \cdots \otimes A^{-\frac{1}{2}}h_1 \quad \text{for all } h_1, \ldots, h_n \in H_{\mathbb{R}} \cap \mathrm{dom}A^{-\frac{1}{2}}$$

The modular group of automorphisms $(\sigma_t)_{t\in\mathbb{R}}$ on $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ relative to φ is given by:

$$\sigma_t(G(f)) = \Delta^{it}G(f)\Delta^{-it} = G(U_{-t}f) \qquad \text{for all} \quad t \in \mathbb{R} \quad \text{and all} \quad f \in H_{\mathbb{R}}$$

In the following Lemma we state a well known formula giving, in particular, all moments of the q-Gaussians.

Lemma 2.1 *Let $r \in \mathbb{N}_*$ and $(h_l)_{\substack{-r\leqslant l\leqslant r\\ k\neq 0}}$ be a family of vectors in $H_{\mathbb{R}}$. For all $l \in \{1, \ldots, r\}$ consider the operator $d_l = a^*(h_l) + a(h_{-l})$. For all $(k(1), \ldots, k(r)) \in \{1, *\}^r$ we have:*

$$\varphi(d_1^{k(1)} \ldots d_r^{k(r)}) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ \displaystyle\sum_{\substack{\mathcal{V} \text{ 2-partition} \\ \mathcal{V}=\{(s_l,t_l)_{l=1}^{l=p}\} \text{with } s_l<t_l}} q^{i(\mathcal{V})} \prod_{l=1}^{p} \varphi(d_{s_l}^{k(s_l)} d_{t_l}^{k(t_l)}) & \text{if } r = 2p \end{cases}$$

where $i(\mathcal{V}) = \#\{(k,l),\ s_k < s_l < t_k < t_l\}$ is the number of crossings of the 2-partition $\mathcal{V}$.

Remarks.

• When $(U_t)_{t\in\mathbb{R}}$ is trivial, $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ reduces to Bożejko and Speicher's q-Gaussian algebra $\Gamma_q(H_\mathbb{R})$. This is the only case where φ is a trace on $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$. Actually, $\Gamma_q(H_\mathbb{R})$ is known to be a non-hyperfinite II_1 factor (see [BKS], [BS1], [Nou] and [Ri]). In all other cases $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ turns out to be a type *III* von Neumann algebra (see [Sh] and [Hi]).

• Lemma 2.1 implies that for all $n\in\mathbb{N}$ and all $f\in H_\mathbb{R}$:

$$\varphi(G(f)^{2n}) = \sum_{\mathcal{V}\text{ 2-partition}} q^{i(\mathcal{V})}\|f\|^{2n}_{H_\mathbb{R}}$$

Therefore, we see that the distribution of a single gaussian does not depend on the group $(U_t)_{t\in\mathbb{R}}$. In the tracial case (thus in all cases), and when $\|f\|=1$, this distribution is the absolutely continuous probability measure ν_q supported on the interval $[-2/\sqrt{1-q},2/\sqrt{1-q}]$ whose orthogonal polynomials are the q-Hermite polynomials (see [BKS]). In particular, we have :

$$\text{For all}\quad f\in H_\mathbb{R},\quad \|G(f)\| = \frac{2}{\sqrt{1-q}}\|f\|_{H_\mathbb{R}} \tag{1}$$

2.2 The finite dimensional case

We now briefly recall a description of the von Neumann algebra $\Gamma_q(H_\mathbb{R},U_t)$ where $H_\mathbb{R}$ is an Euclidian space of dimension $2k$ ($k\in\mathbb{N}_*$). There exists $(H_j)_{1\leqslant j\leqslant k}$ a family of two dimensional spaces, invariant under $(U_t)_{t\in\mathbb{R}}$, and $(\lambda_j)_{1\leqslant j\leqslant k}$ some real numbers greater or equal to 1 such that for all $j\in\{1,\dots,k\}$,

$$H_\mathbb{R} = \underset{1\leqslant j\leqslant k}{\oplus} H_j \quad\text{and}\quad U_{t|H_j} = \begin{pmatrix} \cos(t\ln(\lambda_j)) & -\sin(t\ln(\lambda_j)) \\ \sin(t\ln(\lambda_j)) & \cos(t\ln(\lambda_j)) \end{pmatrix}$$

We put $I=\{-k,\dots,-1\}\cup\{1,\dots,k\}$. It is then easily checked that the deformed scalar product $\langle\,.\,,\,.\,\rangle_U$ on the complexification $H_\mathbb{C}$ of $H_\mathbb{R}$ is characterized by the condition that there exists a basis $(f_j)_{j\in I}$ in $H_\mathbb{R}$ such that for all $(j,l)\in\{1,\dots,k\}^2$

$$\langle f_j, f_{-l}\rangle_U = \delta_{j,l}.i\frac{\lambda_j-1}{\lambda_j+1} \quad\text{and}\quad \langle f_{\pm j}, f_{\pm l}\rangle_U = \delta_{j,l} \tag{2}$$

For all $j\in\{1,\dots,k\}$ we put $\mu_j=\lambda_j^{\frac{1}{4}}$. Let $(e_j)_{j\in I}$ be a real orthonormal basis of $\mathbb{C}^{2k}$ equipped with its canonical scalar product. For all $j\in\{1,\dots,k\}$ we put

$$\hat{f}_j = \frac{1}{\sqrt{\mu_j^2+\mu_j^{-2}}}\left(\mu_j e_{-j} + \mu_j^{-1}e_j\right) \quad\text{and}\quad \hat{f}_{-j} = \frac{i}{\sqrt{\mu_j^2+\mu_j^{-2}}}\left(\mu_j e_{-j} - \mu_j^{-1}e_j\right)$$

It is easy to see that the conditions (2) are fulfilled for the family $(\hat{f}_j)_{j\in I}$. We will denote by $H_\mathbb{R}$ the Euclidian space generated by the family $(\hat{f}_j)_{j\in I}$ in $\mathbb{C}^{2k}$. This provides us with a realization of $\Gamma_q(H_\mathbb{R},U_t)$ as a subalgebra of $B\left(\mathcal{F}_q(\mathbb{C}^{2k})\right)$. Indeed, $\Gamma_q(H_\mathbb{R},U_t)=\{G(\hat{f}_j),\, j\in I\}''\subset B\left(\mathcal{F}_q(\mathbb{C}^{2k})\right)$. For all $j\in\{1,\dots,k\}$ put

$$f_j = \frac{\sqrt{\mu_j^2+\mu_j^{-2}}}{2}\hat{f}_j \quad\text{and}\quad f_{-j} = \frac{\sqrt{\mu_j^2+\mu_j^{-2}}}{2}\hat{f}_{-j}$$

We define the following generalized semi-circular variable by:

$$c_j = G(f_j) + iG(f_{-j}) = W(f_j + if_{-j})$$

It is clear that $\Gamma_q(H_{\mathbb{R}}, U_t) = \{c_j,\ j \in \{1, \ldots, k\}\}'' \subset B\left(\mathcal{F}_q(\mathbb{C}^{2k})\right)$ and we can check that

$$c_j = \mu_j a(e_{-j}) + \mu_j^{-1} a^*(e_j) \tag{3}$$

Moreover, for all $j \in \{1, \ldots, k\}$, c_j is an entire vector for $(\sigma_t)_{t\in\mathbb{R}}$ and we have, for all $z \in \mathbb{C}$:

$$\sigma_z(c_j) = \lambda_j^{iz} c_j.$$

Recall that all odd $*$-moments of the family $(c_j)_{1\leqslant j\leqslant k}$ are zero. Applying Lemma 2.1 to the operators c_j we state, for further references, an explicit formula for the $*$-moments of $(c_j)_{1\leqslant j\leqslant k}$. In the following we use the convention $c^{-1} = c^*$ when there is no possible confusion.

Lemma 2.2 *Let $r \in \mathbb{N}_*$, $(j(1), \ldots, j(2r)) \in \{1, \ldots, k\}^{2r}$ and $(k(1), \ldots, k(2r)) \in \{\pm 1\}^{2r}$*

$$\begin{aligned}
\varphi(c_{j(1)}^{k(1)} \ldots c_{j(2r)}^{k(2r)}) &= \sum_{\substack{\mathcal{V} \text{ 2-partition} \\ \mathcal{V}=\{(s_l,t_l)_{l=1}^{l=r}\} \text{with } s_l<t_l}} q^{i(\mathcal{V})} \prod_{l=1}^{r} \varphi(c_{j(s_l)}^{k(s_l)} c_{j(t_l)}^{k(t_l)}) \\
&= \sum_{\substack{\mathcal{V} \text{ 2-partition} \\ \mathcal{V}=\{(s_l,t_l)_{l=1}^{l=r}\} \text{with } s_l<t_l}} q^{i(\mathcal{V})} \prod_{l=1}^{r} \mu_{j(s_l)}^{2k(s_l)}\, \delta_{k(s_l),\,-k(t_l)}\, \delta_{j(s_l),j(t_l)}
\end{aligned}$$

Proof. As said above this is a consequence of Lemma 2.1 and the explicit computation of covariances. Using (3) we have:

$$\begin{aligned}
\varphi(c_{j(1)}^{k(1)} c_{j(2)}^{k(2)}) &= \langle c_{j(1)}^{-k(1)}\Omega, c_{j(2)}^{k(2)}\Omega\rangle \\
&= \langle \mu_{j(1)}^{k(1)} e_{-k(1)j(1)}, \mu_{j(2)}^{-k(2)} e_{k(2)j(2)}\rangle \\
&= \mu_{j(1)}^{2k(1)}\, \delta_{k(1),\,-k(2)}\, \delta_{j(1),j(2)}
\end{aligned}$$

□

2.3 Baby Fock

The symmetric Baby Fock (also known as symmetric toy Fock space) is at some point a discrete approximation of the bosonic Fock space (see [PAM]). In [Bi], Biane considered spin systems with mixed commutation and anti-commutation relations (which is a generalization of the symmetric toy Fock), and used it to approximate q-Fock space (via Speicher central limit Theorem). In this section we recall the formal construction of [Bi]. Let I be a finite subset of $\mathbb{Z}$ and ϵ a function from $I \times I$ to $\{-1, 1\}$ satisfying for all $(i, j) \in I^2$, $\epsilon(i, j) = \epsilon(j, i)$ and $\epsilon(i, i) = -1$. Let $\mathcal{A}(I, \epsilon)$ be the free complex unital algebra with generators $(x_i)_{i\in I}$ quotiented by the relations

$$x_i x_j - \epsilon(i, j) x_j x_i = 2\delta_{i,j} \quad \text{for} \quad (i, j) \in I^2 \tag{4}$$

We define an involution on $\mathcal{A}(I, \epsilon)$ by $x_i^* = x_i$. For a subset $A = \{i_1, \ldots, i_k\}$ of I with $i_1 < \cdots < i_k$ we put $x_A = x_{i_1} \ldots x_{i_k}$, where, by convention, $x_\varnothing = 1$. Then $(x_A)_{A\subset I}$ is a basis of the vector space $\mathcal{A}(I,\epsilon)$. Let φ^ϵ be the tracial functional defined by $\varphi^\epsilon(x_A) = \delta_{A,\varnothing}$ for all $A \subset I$. $\langle x, y\rangle = \varphi^\epsilon(x^* y)$ defines a positive definite hermitian form on $\mathcal{A}(I,\epsilon)$. We will denote by $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$ the Hilbert space $\mathcal{A}(I,\epsilon)$ equipped with $\langle\, .\, ,\, .\,\rangle$. $(x_A)_{A\subset I}$ is an orthonormal basis of $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$. For each $i \in I$, define the following partial isometries β_i^* and α_i^* of $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$ by:

$$\beta_i^*(x_A) = \begin{cases} x_i x_A & \text{if} \quad i \notin A \\ 0 & \text{if} \quad i \in A \end{cases} \quad \text{and} \quad \alpha_i^*(x_A) = \begin{cases} x_A x_i & \text{if} \quad i \notin A \\ 0 & \text{if} \quad i \in A \end{cases}$$

Note that their adjoints are given by:

$$\beta_i(x_A) = \begin{cases} x_i x_A & \text{if} \quad i \in A \\ 0 & \text{if} \quad i \notin A \end{cases} \quad \text{and} \quad \alpha_i(x_A) = \begin{cases} x_A x_i & \text{if} \quad i \in A \\ 0 & \text{if} \quad i \notin A \end{cases}$$

β_i^* and β_i (respectively α_i^* and α_i) are called the left (respectively right) creation and annihilation operators at the Baby Fock level. In the next Lemma we recall from [Bi] the fundamental relations 1. and 2., and we leave the proof of 3., 4. and 5. to the reader.

Lemma 2.3 *The following relations hold:*

1. *For all* $i \in I$ $(\beta_i^*)^2 = \beta_i^2 = 0$ *and* $\beta_i\beta_i^* + \beta_i^*\beta_i = Id$.
2. *For all* $(i,j) \in I^2$ *with* $i \neq j$ $\beta_i\beta_j - \epsilon(i,j)\beta_j\beta_i = 0$ *and* $\beta_i\beta_j^* - \epsilon(i,j)\beta_j^*\beta_i = 0$.
3. *Same relations as in 1. and 2. with* α *in place of* β.
4. *For all* $i \in I$ $\beta_i^*\alpha_i^* = \alpha_i^*\beta_i^* = 0$ *and for all* $(i,j) \in I^2$ *with* $i \neq j$ $\beta_i^*\alpha_j^* = \alpha_j^*\beta_i^*$.
5. *For all* $(i,j) \in I^2$ $\beta_i^*\alpha_j = \alpha_j\beta_i^*$.

It is easily seen, by 1. and 2. of Lemma 2.3, that the self adjoint operators defined by: $\gamma_i = \beta_i^* + \beta_i$ satisfy the following relation :

$$\text{for all} \quad (i,j) \in I^2, \quad \gamma_i\gamma_j - \epsilon(i,j)\gamma_j\gamma_i = 2\delta_{i,j}\text{Id} \tag{5}$$

Let $\Gamma_I \subset B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ be the $*$-algebra generated by all γ_i, $i \in I$. Still denoting by φ^ϵ the vector state associated to the vector 1, it is known that φ^ϵ is a faithful normalized trace on the finite dimensional C^*-algebra Γ_I (see the remarks below). Moreover, $\Gamma_I \subset B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ is the faithful GNS representation of $(\Gamma_I,\varphi^\epsilon)$ with cyclic and separating vector 1.

Remarks.

• It is clear that we can do the previous construction for some finite sets I that are not given explicitly as subsets of $\mathbb{Z}$. Then, to each total order on I we can associate a basis $(x_A)_{A\subset I}$ of $\mathcal{A}(I,\epsilon)$. But, because of the commutation relations (4), the state φ^ϵ, the scalar product on $\mathcal{A}(I,\epsilon)$ and the creation operators do not depend on the chosen total order.

• We can also extend the previous construction to not necessarily finite sets I. Only the faithfulness of φ^ϵ on Γ_I requires some comments. It suffices to see that the vector 1 is separating for Γ_I. Indeed, set $\delta_i = \alpha_i^* + \alpha_i$ for $i \in I$, and

$$\Gamma_{r,I} = \{\delta_i, i \in I\}'' \subset B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)).$$

Then, it is clear from 4. and 5. of Lemma 2.3, that $\Gamma_{r,I} \subset \Gamma'_I$ (there is actually equality). Since 1 is clearly cyclic for $\Gamma_{r,I}$, then it is also cyclic for Γ'_I, thus 1 is separating for Γ_I.

• Let I and J, $I \subset J$, be some sets together with signs ϵ and ϵ' such that $\epsilon'_{|I\times I} = \epsilon$. It is clear that $L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)$ embeds isometrically in $L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'})$. Set $K = J \setminus I$. Fix some total orders on I and K and consider the total order on J which coincides with the orders of I and K and such that any element of I is smaller than any element of K. The associated orthonormal basis of $L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'})$ is given by the family $(x_A x_B)_{A\in\mathcal{F}(I),B\in\mathcal{F}(K)}$ (where $\mathcal{F}(I)$, respectively $\mathcal{F}(K)$, denotes the set of finite subsets of I, respectively K). In particular with have the following Hilbertian decomposition:

$$L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}) = \bigoplus_{B\in\mathcal{F}(K)} L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon)x_B \tag{6}$$

For $j \in I$ we (temporarily) denote by $\widetilde{\beta}_j$ the annihilation operator in $B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ and simply by β_j its analogue in $B(L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}))$. Let $\widetilde{C}_I$ (respectively C_J) be the C^*-algebra generated by $\{\widetilde{\beta}_j,\ j\in I\}$ (respectively $\{\beta_j,\ j\in J\}$) in $B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ (respectively $B(L^2(\mathcal{A}(J,\epsilon'),\varphi^{\epsilon'}))$). Consider also C_I the C^*-algebra generated by $\{\beta_j,\ j\in I\}$ in $B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$. For $B = \{j_1,\dots,j_k\} \subset K$, with $j_1 < \dots < j_k$, let us denote by α_B the operator $\alpha_{j_1}\dots\alpha_{j_k}$. If $\widetilde{T} \in \widetilde{C}_I$ and if T denotes its counterpart in C_I, then it is easily seen that, with respect to the Hilbertian decomposition (6), we have

$$T = \bigoplus_{B\in\mathcal{F}(K)} \alpha_B^* \widetilde{T}\alpha_B. \tag{7}$$

It follows that $\widetilde{C}_I$ is $*$-isomorphic to $C_I \subset C_J$.

• It is possible to find explicitly selfadjoint matrices satisfying the mixed commutation and anti-commutation relations (5) (see [Sp] and [Bi]). We choose to present this approach because it will be easier to handle the objects of modular theory in this abstract situation when we will deal with non-tracial von Neumann algebras (see section 5).

2.4 Speicher's central limit Theorem

We recall Speicher's central limit theorem which is specially designed to handle either commuting or anti-commuting (depending on a function ϵ) independent variables. Roughly speaking, Speicher's central limit theorem asserts that such a family of centered noncommutative variables which have a fixed covariance, and uniformly bounded $*$-moments, is convergent in $*$-moments, as soon as a combinatorial quantity associated with ϵ is converging. Moreover the limit $*$-distribution is only determined by the common covariance and the limit of the combinatorial quantity.

We start by recalling some basic notions on independence and set partitions.

Definition 2.4 *Let $(\mathcal{A},\varphi)$ be a $*$-algebra equipped with a state φ and $(\mathcal{A}_i)_{i\in I}$ a family of C^*-subalgebras of $\mathcal{A}$. The family $(\mathcal{A}_i)_{i\in I}$ is said to be independent if for all $r\in\mathbb{N}_*$, $(i_1,\dots,i_r)\in I^r$ with $i_s \neq i_t$ for $s\neq t$, and all $a_{i_s}\in\mathcal{A}_{i_s}$ for $s\in\{1,\dots,r\}$ we have:*

$$\varphi(a_{i_1}\dots a_{i_r}) = \varphi(a_{i_1})\dots\varphi(a_{i_r})$$

As usual, a family $(a_i)_{i\in I}$ of non-commutative random variables of $\mathcal{A}$ will be called independent if the family of C^*-subalgebras of $\mathcal{A}$ that they generate is independent.

On the set of p-uples of integers belonging to $\{1,\ldots,N\}$ define the equivalence relation $\sim$ by:

$$(i(1),\ldots,i(p)) \sim (j(1),\ldots,j(p)) \text{ if } \big(i(l)=i(m) \Longleftrightarrow j(l)=j(m)\big)\ \forall\,(l,m)\in\{1,\ldots,p\}^2$$

Then the equivalence classes for the relation $\sim$ are given by the partitions of the set $\{1,\ldots,p\}$. We denote by $V_1,\ldots,V_r$ the blocks of the partition $\mathcal{V}$ and we call $\mathcal{V}$ a 2-partition if each of these blocks is of cardinal 2. The set of all 2-partitions of the set $\{1,\ldots,p\}$ (p even) will be denoted by $\mathcal{P}_2(1,\ldots,p)$. For $\mathcal{V}\in\mathcal{P}_2(1,\ldots,2r)$ let us denote by $V_l=(s_l,t_l)$, $s_l<t_l$, for $l\in\{1,\ldots,r\}$ the blocks of the partition $\mathcal{V}$. The set of crossings of $\mathcal{V}$ is defined by

$$I(\mathcal{V})=\{(l,m)\in\{1,\ldots,r\}^2,\ s_l<s_m<t_l<t_m\}$$

The 2-partition $\mathcal{V}$ is said to be crossing if $I(\mathcal{V})\neq\varnothing$ and non-crossing if $I(\mathcal{V})=\varnothing$.

Theorem 2.5 (Speicher) *Consider k sequences $(b_{i,j})_{(i,j)\in\mathbb{N}_*\times\{1,\ldots,k\}}$ in a non-commutative probability space (B,φ) satisfying the following conditions:*

1. *The family $(b_{i,j})_{(i,j)\in\mathbb{N}_*\times\{1,\ldots,k\}}$ is independent.*
2. *For all $(i,j)\in\mathbb{N}_*\times\{1,\ldots,k\}$, $\varphi(b_{i,j})=0$*
3. *For all $(k(1),k(2))\in\{-1,1\}^2$ and $(j(1),j(2))\in\{1,\ldots,k\}^2$, the covariance $\varphi(b_{i,j(1)}^{k(1)}b_{i,j(2)}^{k(2)})$ is independent of i and will be denoted by $\varphi(b_{j(1)}^{k(1)}b_{j(2)}^{k(2)})$.*
4. *For all $w\in\mathbb{N}_*$, $(k(1),\ldots,k(w))\in\{-1,1\}^w$ and all $j\in\{1,\ldots,k\}$ there exists a constant C such that for all $i\in\mathbb{N}_*$, $|\varphi(b_{i,j}^{k(1)}\ldots b_{i,j}^{k(w)})|\leqslant C$.*
5. *For all $(i(1),i(2))\in\mathbb{N}_*^2$ there exists a sign $\epsilon(i(1),i(2))\in\{-1,1\}$ such that for all $(j(1),j(2))\in\{1,\ldots,k\}^2$ with $(i(1),j(1))\neq(i(2),j(2))$ and all $(k(1),k(2))\in\{-1,1\}^2$ we have*
$$b_{i(1),j(1)}^{k(1)}b_{i(2),j(2)}^{k(2)}-\epsilon(i(1),i(2))b_{i(2),j(2)}^{k(2)}b_{i(1),j(1)}^{k(1)}=0.$$
(notice that the function ϵ is necessarily symmetric in its two arguments).
6. *For all $r\in\mathbb{N}_*$ and all $\mathcal{V}=\{(s_l,t_l)_{l=1}^{l=r}\}\in\mathcal{P}_2(1,\ldots,2r)$ the following limit exists*
$$t(\mathcal{V})=\lim_{N\to+\infty}\frac{1}{N^r}\sum_{\substack{i(s_1),\ldots,i(s_r)=1\\ i(s_l)\neq i(s_m)\ for\ l\neq m}}^{N}\ \prod_{(l,m)\in I(\mathcal{V})}\epsilon\left(i(s_l),i(s_m)\right))$$

Let $S_{N,j}=\dfrac{1}{\sqrt{N}}\sum_{i=1}^N b_{i,j}$. Then we have for all $p\in\mathbb{N}_*$, $(k(1),\ldots,k(p))\in\{-1,1\}^p$ and all $(j(1),\ldots,j(p))\in\{1,\ldots,k\}^p$:

$$\lim_{N\to+\infty}\varphi(S_{N,j(1)}^{k(1)}\ldots S_{N,j(p)}^{k(p)})=\begin{cases}0 & \text{if } p \text{ is odd}\\[2ex] \displaystyle\sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\ldots,2r)\\ \mathcal{V}=\{(s_l,t_l)_{l=1}^{l=r}\}}} t(\mathcal{V})\prod_{l=1}^{r}\varphi(b_{j(s_l)}^{k(s_l)}b_{j(t_l)}^{k(t_l)}) & \text{if } p=2r\end{cases}$$

Remark. Speicher's Theorem is proved in [Sp] for a single limit variable. One could either convince oneself that the proof of Theorem 2.5 goes along the same lines, or deduce it from Speicher's usual theorem. Indeed, it suffices to apply Speicher's theorem to the family $\left(\sum_{j=1}^{k} z_j b_{i,j}\right)_{i\in\mathbb{N}}$, for all $(z_1,\ldots,z_k)\in\mathbb{T}^k$ and to identify the Fourier coefficients of the limit $*$-moments.

The following Lemma, proved in [Sp], guarantees the almost sure convergence of the quantity $t(\mathcal{V})$ provided that the function ϵ has independent entries following the same 2-points Dirac distribution:

Lemma 2.6 *Let $q\in(-1,1)$ and consider a family of random variables $\epsilon(i,j)$ for $(i,j)\in\mathbb{N}_*$ with $i\neq j$, such that*

1. *For all $(i,j)\in\mathbb{N}_*$ with $i\neq j$, $\epsilon(i,j)=\epsilon(j,i)$*
2. *The family $(\epsilon(i,j))_{i>j}$ is independent*
3. *For all $(i,j)\in\mathbb{N}_*$ with $i\neq j$ the probability distribution of $\epsilon(i,j)$ is*

$$\frac{1+q}{2}\delta_1+\frac{1-q}{2}\delta_{-1}$$

Then, almost surely, we have for all $r\in\mathbb{N}_*$ and for all $\mathcal{V}\in\mathcal{P}_2(1,\ldots,2r)$

$$\lim_{N\to+\infty}\frac{1}{N^r}\sum_{\substack{i(s_1),\ldots,i(s_r)=1\\ i(s_l)\neq i(s_m)\ for\ l\neq m}}^{N}\ \prod_{(l,m)\in I(\mathcal{V})}\epsilon(i(s_l),i(s_m))=q^{i(\mathcal{V})}$$

Remark. It is now a straightforward verification to see that Theorem 2.5 combined with Lemma 2.6 can be applied to families of mixed commuting /anti-commuting Gaussian operators (see Lemma 5.4 for the independence condition). The limit moments are those given by the classical q-Gaussian operators (by classical we mean that $(U_t)_{t\in\mathbb{R}}$ is trivial).
Alternatively, one can apply directly Speicher's theorem to families of mixed commuting /anti-commuting creation operators as it is done in [Sp] and [Bi]. The limit $*$-moments are in this case the $*$-moments of classical q-creation operators.

3 The tracial case

Our goal in this section is to show that $\Gamma_q(H_\mathbb{R})$ is QWEP. In fact, by inductive limit, it is sufficient to prove it for $H_\mathbb{R}$ finite dimensional. Let $k\geqslant 1$. We will consider $\mathbb{R}^k$ as the real Hilbert space of dimension k, with the canonical orthonormal basis $(e_1,\ldots,e_k)$, and $\mathbb{C}^k$, its complex counterpart. Let us fix $q\in(-1,1)$ and consider $\Gamma_q(\mathbb{R}^k)$ the von Neumann algebra generated by the q-Gaussians $G(e_1),\ldots,G(e_k)$. We denote by τ the expectation on the vacuum vector, which is a trace in this particular case.

By the ending remark of section 2., there are Hermitian matrices, $g_{n,1}(\omega),\ldots,g_{n,k}(\omega)$, depending on a random parameter denoted by ω and lying in a finite dimensional matrix algebra, such that their joint $*$-distribution converges almost surely to the joint $*$-distribution of the q-Gaussians in the following sense: for all polynomial P in k noncommuting variables,

$$\lim_{n\to\infty} \tau_n(P(g_{n,1}(\omega),\dots,g_{n,k}(\omega))) = \tau(P(G(e_1),\dots,G(e_k))) \quad \text{almost surely in } \omega.$$

We will denote by $\mathcal{A}_n$ the finite dimensional C^*-algebra generated by $g_{n,1}(\omega),\dots,g_{n,k}(\omega)$. We recall that these algebras are equipped with the trace τ_n defined by:

$$\tau_n(x) = \langle 1, x.1\rangle$$

Since the set of all monomials in k noncommuting variables is countable, we have for almost all ω,

$$\lim_{n\to\infty} \tau_n(P(g_{n,1}(\omega),\dots,g_{n,k}(\omega)) = \tau(P(G(e_1),\dots,G(e_k))) \quad \text{for all such monomials } P \tag{8}$$

A fortiori we can find an ω_0 such that (8) holds for ω_0. We will fix such an ω_0 and simply denote by $g_{n,i}$ the matrix $g_{n,i}(\omega_0)$ for all $i \in \{1,\dots,k\}$. With these notations, it is clear that, by linearity, we have for all polynomials P in k noncommuting variables,

$$\lim_{n\to\infty} \tau_n(P(g_{n,1},\dots,g_{n,k})) = \tau(P(G(e_1),\dots,G(e_k))). \tag{9}$$

We need to have a uniform control on the norms of the matrices $g_{n,i}$. Let C be such that $\|G(e_1)\| < C$, we will replace the $g_{n,i}$'s by their truncations $\chi_{]-C,C[}(g_{n,i})g_{n,i}$ (where $\chi_{]-C,C[}$ denotes the characteristic function of the interval $]-C,C[$). For simplicity $\chi_{]-C,C[}(g_{n,i})g_{n,i}$ will be denoted by $\tilde{g}_{n,i}$. We now check that (9) is still valid for the $\tilde{g}_{n,i}$'s.

Lemma 3.1 *With the notations above, for all polynomials P in k noncommuting variables we have*

$$\lim_{n\to\infty} \tau_n(P(\tilde{g}_{n,1},\dots,\tilde{g}_{n,k})) = \tau(P(G(e_1),\dots,G(e_k))). \tag{10}$$

Proof. We just have to prove that for all monomials P in k noncommuting variables we have

$$\lim_{n\to\infty} \tau_n\left[P(\tilde{g}_{n,1},\dots,\tilde{g}_{n,k}) - P(g_{n,1},\dots,g_{n,k})\right] = 0.$$

Writing $g_{n,i} = \tilde{g}_{n,i} + (g_{n,i} - \tilde{g}_{n,i})$ and developing using multilinearity, we are reduced to showing that the L^1-norms of any monomial in $\tilde{g}_{n,i}$ and $(g_{n,i} - \tilde{g}_{n,i})$ (with at least one factor $(g_{n,i} - \tilde{g}_{n,i})$) tend to 0. By the Hölder inequality and the uniform boundedness of the $\|\tilde{g}_{n,i}\|$'s, it suffices to show that for all $i \in \{1 \dots k\}$,

$$\lim_{n\to\infty} \tau_n(|\tilde{g}_{n,i} - g_{n,i}|^p) = 0 \quad \text{for all } p \geqslant 1. \tag{11}$$

Let us prove (11) for $i = 1$. We are now in a commutative setting. Indeed, let us introduce the spectral resolutions of identity, E^n_t (respectively E_t), of $g_{n,1}$ (respectively $G(e_1)$). By (9) we have for all polynomials P

$$\lim_{n\to\infty} \tau_n(P(g_{n,1})) = \tau(P(G(e_1))).$$

We can rewrite this as follows: for all polynomials P

$$\lim_{n\to\infty} \int_{\sigma(g_{n,1})} P(t)\mathrm{d}\langle E^n_t.1, 1\rangle = \int_{\sigma(G(e_1))} P(t)\mathrm{d}\langle E_t.\Omega, \Omega\rangle.$$

Let μ_n (respectively μ) denote the compactly supported probability measure $\langle E_t^n.1, 1\rangle$ (respectively $\langle E_t.\Omega, \Omega\rangle$) on $\mathbb{R}$. With these notations our assumption becomes: for all polynomials P

$$\lim_{n\to\infty} \int P \mathrm{d}\mu_n = \int P \mathrm{d}\mu. \tag{12}$$

and (11) is equivalent to:

$$\lim_{n\to\infty} \int_{|t|\geqslant C} |t|^p \mathrm{d}\mu_n = 0 \quad \text{for all } p \geqslant 1. \tag{13}$$

Then the result follows from the following elementary Lemma. We give a proof for sake of completeness.

Lemma 3.2 *Let $(\mu_n)_{n\geqslant 1}$ be a sequence of compactly supported probability measures on $\mathbb{R}$ converging in moments to a compactly supported probability measure μ on $\mathbb{R}$. Assume that the support of μ is included in the open interval $]-C, C[$. Then,*

$$\lim_{n\to\infty} \int_{|t|\geqslant C} \mathrm{d}\mu_n = 0.$$

Moreover, let f be a borelian function on $\mathbb{R}$ such that there exist $M > 0$ and $r \in \mathbb{N}$ satisfying $|f(t)| \leqslant M(t^{2r}+1)$ for all $t \geqslant C$. Then,

$$\lim_{n\to\infty} \int_{|t|\geqslant C} f \mathrm{d}\mu_n = 0.$$

Proof. For the first assertion, let $C' < C$ such that the support of μ is included in $]-C', C'[$. Let $\epsilon > 0$ and an integer k such that $\left(\frac{C'}{C}\right)^{2k} \leqslant \epsilon$. Let $P(t) = \left(\frac{t}{C}\right)^{2k}$. It is clear that $\chi_{\{|t|\geqslant C\}}(t) \leqslant P(t)$ for all $t \in \mathbb{R}$ and that $\sup_{|t|<C'} P(t) \leqslant \epsilon$. Thus,

$$0 \leqslant \limsup_{n\to\infty} \int_{|t|\geqslant C} \mathrm{d}\mu_n \leqslant \lim_{n\to\infty} \int P(t) \mathrm{d}\mu_n = \int P(t) \mathrm{d}\mu \leqslant \epsilon.$$

Since ϵ is arbitrary, we get $\lim\limits_{n\to\infty} \int_{|t|\geqslant C} \mathrm{d}\mu_n = 0$.
The second assertion is a consequence of the first one. Let f be a borelian function on $\mathbb{R}$ such that there exist $M > 0$ and $r \in \mathbb{N}$ satisfying $|f(t)| \leqslant M(t^{2r}+1)$ for all $t \in \mathbb{R}$. Using the Cauchy-Schwarz inequality we get:

$$\begin{aligned}
0 \leqslant \limsup_{n\to\infty} \int_{|t|\geqslant C} |f| \mathrm{d}\mu_n &\leqslant \limsup_{n\to\infty} \int_{|t|\geqslant C} M(t^{2r}+1) \mathrm{d}\mu_n \\
&\leqslant M \lim_{n\to\infty} \left(\int (t^{2r}+1)^2 \mathrm{d}\mu_n \right)^{\frac{1}{2}} \lim_{n\to\infty} \left(\int_{|t|\geqslant C} \mathrm{d}\mu_n \right)^{\frac{1}{2}} \\
&\leqslant M \left(\int (t^{2r}+1)^2 \mathrm{d}\mu \right)^{\frac{1}{2}} \lim_{n\to\infty} \left(\int_{|t|\geqslant C} \mathrm{d}\mu_n \right)^{\frac{1}{2}} = 0
\end{aligned}$$

□

Remark. Let us define $\mathcal{A}$ as the $*$-algebra generated by $G(e_1), \ldots, G(e_k)$. Observe that $\mathcal{A}$ is isomorphic to the $*$-algebra of all polynomials in k noncommuting variables (the free complex

$*$-algebra with k generators). Indeed, if $P(G(e_1), \ldots, G(e_k)) = 0$ for a polynomial P in k noncommuting variables, then the equation $P(G(e_1), \ldots, G(e_k))\Omega = 0$ implies that all coefficients of monomials of highest degree are 0, and thus $P = 0$ by induction. More generally, this remains true for the q-Araki Woods algebras $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$: if $(e_i)_{i\in I}$ is a free family of vectors in $H_{\mathbb{R}}$ then the $*$-algebra $\mathcal{A}_I$ generated by the family $(G(e_i))_{i\in I}$ is isomorphic to the free complex $*$-algebra with I generators.

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}^*$ and consider the ultraproduct von Neumann algebra (see [P] section 9.10) N defined by

$$N = \left(\prod_{n\geqslant 1} \mathcal{A}_n\right) \Big/ I_{\mathcal{U}}$$

where $I_{\mathcal{U}} = \{(x_n)_{n\geqslant 1} \in \prod\limits_{n\geqslant 1} \mathcal{A}_n,\ \lim\limits_{\mathcal{U}} \tau_n(x_n^* x_n) = 0\}$. The von Neumann algebra N is equipped with the faithful normal and normalized trace $\tau((x_n)_{n\geqslant 1}) = \lim\limits_{\mathcal{U}} \tau_n(x_n)$ (which is well defined).

Using the asymptotic matrix model for the q-Gaussians and by the preceding remark, we can define a $*$-homomorphism φ between the $*$-algebras $\mathcal{A}$ and N in the following way:

$$\varphi(P(G(e_1), \ldots, G(e_k))) = (P(\tilde{g}_{n,1}, \ldots, \tilde{g}_{n,k}))_{n\geqslant 1}$$

for every polynomial P in k noncommuting variables. By Lemma 3.1, φ is trace preserving on $\mathcal{A}$. Since the $*$-algebra $\mathcal{A}$ is weak-$*$ dense in $\Gamma_q(\mathbb{R}^k)$, φ extends naturally to a trace preserving homomorphism of von Neumann algebras, that is still denoted by φ (see Lemma 4.2 below for a more general result). It follows that $\Gamma_q(\mathbb{R}^k)$ is isomorphic to a sub-algebra of N which is the image of a conditional expectation (this is automatic in the tracial case). Since the $\mathcal{A}_n$'s are finite dimensional, they are injective, hence their product is injective and a fortiori has the WEP, and thus N is QWEP. Since $\Gamma_q(\mathbb{R}^k)$ is isomorphic to a sub-algebra of N which is the image of a conditional expectation, $\Gamma_q(\mathbb{R}^k)$ is also QWEP (see [Oz]). We have obtained the following:

Theorem 3.3 *Let $H_{\mathbb{R}}$ be a real Hilbert space and $q \in (-1, 1)$. The von Neumann algebra $\Gamma_q(H_{\mathbb{R}})$ is QWEP.*

Proof. Our previous discussion implies the result for every finite dimensional $H_{\mathbb{R}}$. The general result is a consequence of the stability of QWEP by inductive limit (see [Kir] and [Oz] Proposition 4.1 (iii)). □

Let $C_q^*(H_{\mathbb{R}})$ be the C^*-algebra generated by all q-Gaussians:

$$C_q^*(H_{\mathbb{R}}) = C^*\big(\{G(f), f \in H_{\mathbb{R}}\}\big) \subset B(\mathcal{F}_q(H_{\mathbb{C}})).$$

We now deduce the following strengthening of Theorem 3.3.

Corollary 3.4 *Let $H_{\mathbb{R}}$ be a real Hilbert space and $q \in (-1, 1)$. The C^*-algebra $C_q^*(H_{\mathbb{R}})$ is QWEP.*

Let A, B, with $A \subset B$ be C^*-algebras. Recall (from [Oz]) that A is said to be weakly cp complemented in B, if there exists a unital completely positive map $\Phi : B \longrightarrow A^{**}$ such that $\Phi_{|A} = \mathrm{id}_A$. Corollary 3.4 is then a consequence of the following Lemma.

Lemma 3.5 *The C^*-algebra $C_q^*(H_\mathbb{R})$ is weakly cp complemented in the von Neumann algebra $\Gamma_q(H_\mathbb{R})$.*

Proof. For any $t \in \mathbb{R}_+$ denote by Φ_t the unital completely positive maps which are the second quantization of $e^{-t}\mathrm{id} : H_\mathbb{R} \longrightarrow H_\mathbb{R}$ (see [BKS]):

$$\Phi_t = \Gamma_q(e^{-t}\mathrm{id}) : \Gamma_q(H_\mathbb{R}) \longrightarrow \Gamma_q(H_\mathbb{R}), \quad \text{for all} \quad t \geqslant 0.$$

$(\Phi_t)_{t\in\mathbb{R}_+}$ is a semi-group of unital completely positive maps which is also known as the q-Ornstein-Uhlenbeck semi-group. By the well-known ultracontractivity of the semi-group $(\Phi_t)_{t\in\mathbb{R}_+}$ (see [B]), for all $t \in \mathbb{R}_+^*$ and all $W(\xi) \in \Gamma_q(H_\mathbb{R})$, we have

$$\|\Phi_t(W(\xi))\| \leqslant C_{|q|}^{\frac{3}{2}} \frac{1}{1-e^{-t}} \|\xi\|. \tag{14}$$

On the other hand, as a consequence of the Haagerup-Bożejko's inequality (see [B]), for every $n \in \mathbb{N}$ and for every $\xi_n \in H_\mathbb{C}^{\otimes n}$, we have $W(\xi_n) \in C_q^*(H_\mathbb{R})$. Fix $t \in \mathbb{R}_+^*$, $W(\xi) \in \Gamma_q(H_\mathbb{R})$, and write $\xi = \sum_{n\in\mathbb{N}} \xi_n$ with $\xi_n \in H_\mathbb{C}^{\otimes n}$ for all n. From our last observation, for all $N \in \mathbb{N}$,

$$T_N = \Phi_t\big(W\big(\sum_{n=0}^{N} \xi_n\big)\big) = \sum_{n=0}^{N} e^{-tn} W(\xi_n) \in C_q^*(H_\mathbb{R}).$$

By (14), $\Phi_t(W(\xi))$ is the norm limit of the sequence $(T_N)_{N\in\mathbb{N}}$, so $\Phi_t(W(\xi))$ belongs to $C_q^*(H_\mathbb{R})$. It follows that Φ_t maps $\Gamma_q(H_\mathbb{R})$ into $C_q^*(H_\mathbb{R})$. Moreover, it is clear that

$$\lim_{t\to 0} \|\Phi_t(W(\xi)) - W(\xi)\| = 0, \quad \text{for all} \quad W(\xi) \in C_q^*(H_\mathbb{R}). \tag{15}$$

Take $(t_n)_{n\in\mathbb{N}}$ a sequence of positive real numbers converging to 0 and fix $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$. By w^*-compactness of the closed balls in $\big(C_q^*(H_\mathbb{R})\big)^{**}$, we can define the following mapping $\Phi : \Gamma_q(H_\mathbb{R}) \longrightarrow \big(C_q^*(H_\mathbb{R})\big)^{**}$ by

$$\Phi(W(\xi)) = w^*\text{-}\lim_{n,\mathcal{U}} \Phi_{t_n}(W(\xi)), \quad \text{for all} \quad W(\xi) \in \Gamma_q(H_\mathbb{R}).$$

Φ is a unital completely positive map satisfying $\Phi_{|C_q^*(H_\mathbb{R})} = \mathrm{id}_{C_q^*(H_\mathbb{R})}$ by (15). □

Proof of Corollary 3.4. This is a consequence of Theorem 3.3, Lemma 3.5 and Proposition 4.1 (ii) in [Oz]. □

4 Embedding into an ultraproduct

The general setting is as follows. We start with a family $((\mathcal{A}_n, \varphi_n))_{n\in\mathbb{N}}$ of von Neumann algebras equipped with normal faithful state φ_n. We assume that $\mathcal{A}_n \subset B(H_n)$, where the inclusion is given by the G.N.S. representation of $(\mathcal{A}_n, \varphi_n)$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let

$$\widetilde{\mathcal{A}} = \prod_{n\in\mathbb{N}} \mathcal{A}_n/\mathcal{U}$$

be the C^*-ultraproduct over $\mathcal{U}$ of the algebras $\mathcal{A}_n$. We canonically identify $\widetilde{\mathcal{A}} \subset B(H)$, where $H = \prod_{n\in\mathbb{N}} H_n/\mathcal{U}$ is the ultraproduct over $\mathcal{U}$ of the Hilbert spaces H_n. Following Raynaud (see [Ray]), we define $\mathcal{A}$, the vN-ultraproduct over $\mathcal{U}$ of the von Neumann algebras $\mathcal{A}_n$, as the w^*-closure of $\widetilde{\mathcal{A}}$ in $B(H)$. Then the predual $\mathcal{A}_*$ of $\mathcal{A}$ is isometrically isomorphic to the Banach ultraproduct over $\mathcal{U}$ of the preduals $(\mathcal{A}_n)_*$:

$$\mathcal{A}_* = \prod_{n\in\mathbb{N}} (\mathcal{A}_n)_* /\mathcal{U} \tag{16}$$

Let us denote by φ the normal state on $\mathcal{A}$ associated to $(\varphi_n)_{n\in\mathbb{N}}$. Note that φ is not faithful on $\mathcal{A}$, so we introduce $p \in \mathcal{A}$ the support of the state φ. Recall that for all $x \in \mathcal{A}$ we have $\varphi(x) = \varphi(xp) = \varphi(px)$, and that $\varphi(x) = 0$ for a positive x implies that $pxp = 0$. Denote by $(p\mathcal{A}p, \varphi)$ the induced von Neumann algebra $p\mathcal{A}p \subset B(pH)$ equipped with the restriction of the state φ. For each $n \in \mathbb{N}$, let $(\sigma_t^n)_{t\in\mathbb{R}}$ be the modular group of automorphisms of φ_n with the associated modular operator given by Δ_n. For all $t \in \mathbb{R}$, let $(\Delta_n^{it})^\bullet$ be the associated unitary in $\prod_{n\in\mathbb{N}} B(H_n)/\mathcal{U} \subset B(H)$. Since $(\sigma_t^n)^\bullet_{n\in\mathbb{N}}$ is the conjugation by $(\Delta_n^{it})^\bullet$, it follows that $(\sigma_t^n)^\bullet_{n\in\mathbb{N}}$ extends by w^*-continuity to a group of $*$-automorphisms of $\mathcal{A}$. Let $(\sigma_t)_{t\in\mathbb{R}}$ be the local modular group of automorphisms of $p\mathcal{A}p$. By Raynaud's result (see Theorem 2.1 in [Ray]), $p\mathcal{A}p$ is stable by $(\sigma_t^n)^\bullet_{n\in\mathbb{N}}$ and the restriction of $(\sigma_t^n)^\bullet_{n\in\mathbb{N}}$ to $p\mathcal{A}p$ coincides with σ_t.

In the following, we consider a von Neumann algebra $\mathcal{N} \subset B(K)$ equipped with a normal faithful state ψ. Let $\widetilde{\mathcal{N}}$ be a w^*-dense $*$-subalgebra of $\mathcal{N}$ and Φ a $*$-homomorphism from $\widetilde{\mathcal{N}}$ into $\mathcal{A}$ whose image will be denoted by $\widetilde{\mathcal{B}}$ with w^*-closure denoted by $\mathcal{B}$:

$$\Phi \ : \ \widetilde{\mathcal{N}} \subset \mathcal{N} \subset B(K) \longrightarrow \widetilde{\mathcal{B}} \subset \mathcal{A} \subset B(H) \quad \text{and} \quad \overline{\widetilde{\mathcal{N}}}^{w^*} = \mathcal{N}, \quad \overline{\widetilde{\mathcal{B}}}^{w^*} = \mathcal{B}$$

By a result of Takesaki (see [Tak]) there is a normal conditional expectation from $p\mathcal{A}p$ onto $p\mathcal{B}p$ if and only if $p\mathcal{B}p$ is stable by the modular group of φ (which is here given by Raynaud's results). Under this condition there will be a normal conditional expectation from $\mathcal{A}$ onto $p\mathcal{B}p$ and $p\mathcal{B}p$ will inherit some of the properties of $\mathcal{A}$. We would like to pull back these properties to $\mathcal{N}$ itself. It turns out that, with good assumptions on Φ (see Lemma 4.1 below), the compression from $\mathcal{B}$ onto $p\mathcal{B}p$ is a $*$-homomorphism. If in addition, we suppose that Φ is state preserving, then $p\Phi p$ can be extended into a w^*-continuous $*$-isomorphism between $\mathcal{N}$ and $p\mathcal{B}p$.

Lemma 4.1 *In the following,* 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3. $\Longleftrightarrow$ 4. $\Longleftrightarrow$ 5.*:*

1. *For all $x \in \widetilde{\mathcal{B}}$ there is a representative $(x_n)_{n\in\mathbb{N}}$ of x such that for all $n \in \mathbb{N}$, x_n is entire for $(\sigma_t^n)_{t\in\mathbb{R}}$ and $(\sigma_{-i}^n(x_n))_{n\in\mathbb{N}}$ is uniformly bounded.*

2. *For all $x \in \widetilde{\mathcal{B}}$ there exists $z \in \mathcal{A}$ such that for all $y \in \mathcal{A}$ we have $\varphi(xy) = \varphi(yz)$.*

3. *For all $(x, y) \in \mathcal{B}^2$: $\varphi(xpy) = \varphi(xy)$*

4. *For all $(x, y) \in \mathcal{B}^2$, $pxyp = pxpyp$, i.e the canonical application from $\mathcal{B}$ to $p\mathcal{B}p$ is a $*$-homomorphism.*

5. *$p \in \mathcal{B}'$.*

Proof. 1. $\Longrightarrow$ 2. Consider $x \in \widetilde{\mathcal{B}}$ with a representative $(x_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, x_n is entire for $(\sigma_t^n)_{t\in\mathbb{R}}$ and $(\sigma_{-i}^n(x_n))_{n\in\mathbb{N}}$ is uniformly bounded. Denote by $z \in \mathcal{A}$ the class $(\sigma_{-i}^n(x_n))_{n\in\mathbb{N}}^\bullet$. By w^*-density and continuity if suffices to consider an element y in $\widetilde{\mathcal{A}}$ with representative $(y_n)_{n\in\mathbb{N}}$. Then,

$$\varphi(xy) = \lim_{n,\mathcal{U}} \varphi_n(x_n y_n) = \lim_{n,\mathcal{U}} \varphi_n(y_n \sigma_{-i}^n(x_n)) = \varphi(yz)$$

2. $\Longrightarrow$ 3. Here again it suffices to consider $(x,y) \in \widetilde{\mathcal{B}}^2$. By assumption there exists $z \in \mathcal{A}$ such that for all $t \in \mathcal{A}$, $\varphi(xt) = \varphi(tz)$. Applying our assumption for $t = py$ and $t = y$ successively, we obtain the desired result:

$$\varphi(xpy) = \varphi(pyz) = \varphi(yz) = \varphi(xy)$$

3. $\Longrightarrow$ 4. Let $x \in \mathcal{B}$. We have, by 3.: $\varphi(x(1-p)x^*) = 0$. Since p is the support of φ and $x(1-p)x^* \geqslant 0$, this implies $px(1-p)x^*p = 0$. Thus for all $x \in \mathcal{B}$ we have

$$pxpx^*p = pxx^*p$$

We conclude by polarization.
4. $\Longrightarrow$ 5. Let q be an orthogonal projection in $\mathcal{B}$. By 4., pqp is again an orthogonal projection and we claim that this is equivalent to $pq = qp$. Indeed, let us denote by x the contraction qp. Then $x^*x = pqp$ and since pqp is an orthogonal projection we have $|x| = pqp$. It follows that the polar decomposition of x is of the form $x = upqp$, with u a partial isometry. Computing x^2, we see that x is a projection:

$$x^2 = upqp(qp) = upqp = x.$$

Since x is contractive, we deduce that x is an orthogonal projection and that $x^* = x$. Thus $pq = qp$. Since $\mathcal{B}$ is generated by its projections, we have $p \in \mathcal{B}'$.
5. $\Longrightarrow$ 3. This is clear. $\square$

We assume that one of the technical conditions of the previous Lemma is fulfilled. Let us denote by $\Theta = p\Phi p$. Θ is a $*$-homomorphism from $\widetilde{\mathcal{N}}$, into $p\mathcal{A}p$.

$$\Theta = p\Phi p \ : \ \widetilde{\mathcal{N}} \longrightarrow p\mathcal{A}p \subset B(pH)$$

We assume that Φ, and hence Θ, is state preserving. Then Θ can be extended into a (w^*-continuous) $*$-isomorphism from $\mathcal{N}$ onto $p\mathcal{B}p$. This is indeed a consequence of the following well known fact:

Lemma 4.2 *Let $(\mathcal{M},\varphi)$ and $(\mathcal{N},\psi)$ be von Neumann algebras equipped with normal faithful states. Let $\widetilde{\mathcal{M}}$, (respectively $\widetilde{\mathcal{N}}$), be a w^* dense $*$-subalgebra of $\mathcal{M}$ (respectively $\mathcal{N}$). Let Ψ be a $*$-homomorphism from $\widetilde{\mathcal{M}}$ onto $\widetilde{\mathcal{N}}$ such that for all $m \in \widetilde{\mathcal{M}}$ we have $\psi(\Psi(m)) = \varphi(m)$ (Ψ is state preserving). Then Ψ extends uniquely into a normal $*$-isomorphism between $\mathcal{M}$ and $\mathcal{N}$.*

Proof. Since φ is faithful, we have for all $m \in \mathcal{M}$, $\|m\| = \lim\limits_{n\to+\infty} \varphi\left((m^*m)^n\right)^{\frac{1}{2n}}$. Thus, since Ψ is state preserving, Ψ is isometric from $\widetilde{\mathcal{M}}$ onto $\widetilde{\mathcal{N}}$. We put

$$\varphi\widetilde{\mathcal{M}} = \{\varphi.m,\ m \in \widetilde{\mathcal{M}}\} \subset \mathcal{M}_* \quad \text{and} \quad \psi\widetilde{\mathcal{N}} = \{\psi.n,\ n \in \widetilde{\mathcal{N}}\} \subset \mathcal{N}_*.$$

$\varphi\widetilde{\mathcal{M}}$ (respectively $\psi\widetilde{\mathcal{N}}$) is dense in $\mathcal{M}_*$ (respectively $\mathcal{N}_*$). Let us define the following linear operator Ξ from $\psi\widetilde{\mathcal{N}}$ onto $\varphi\widetilde{\mathcal{M}}$:

$$\Xi(\psi.\Psi(m)) = \varphi.m \quad \text{for all } m \in \widetilde{\mathcal{M}}$$

Using Kaplansky's density Theorem and the fact that Ψ is isometric, we compute:

$$\begin{aligned}\|\Xi(\psi.\Psi(m)\| &= \sup_{m_0\in\widetilde{\mathcal{M}},\ \|m_0\|\leqslant 1} \|\varphi(mm_0)\| = \sup_{m_0\in\widetilde{\mathcal{M}},\ \|m_0\|\leqslant 1} \|\psi(\Psi(m)\Psi(m_0))\| \\ &= \sup_{n_0\in\widetilde{\mathcal{N}},\ \|n_0\|\leqslant 1} \|\psi(\Psi(m)n_0))\| = \|\psi.\Psi(m)\|\end{aligned}$$

So that Ξ extends into a surjective isometry from $\mathcal{N}_*$ onto $\mathcal{M}_*$. Moreover Ξ is the preadjoint of Ψ. Indeed we have for all $(m, m_0) \in \widetilde{\mathcal{M}}^2$:

$$\langle \psi.\Psi(m), \Psi(m_0)\rangle = \psi(\Psi(m)\Psi(m_0)) = \varphi(mm_0) = \langle \Xi(\psi.\Psi(m)), m_0\rangle$$

Thus Ψ extends to a normal $*$-isomorphism between $\mathcal{N}$ and $\mathcal{M}$. □

In the following Theorem, we sum up what we have proved in the previous discussion:

Theorem 4.3 *Let $(\mathcal{N}, \psi)$ and $(\mathcal{A}_n, \varphi_n)$, for $n \in \mathbb{N}$, be von Neumann algebras equipped with normal faithful states. Let $\mathcal{U}$ be a non trivial ultrafilter on $\mathbb{N}$, and $\mathcal{A}$ the von Neumann algebra ultraproduct over $\mathcal{U}$ of the $\mathcal{A}_n$'s. For all $n \in \mathbb{N}$ let us denote by $(\sigma_t^n)_{t\in\mathbb{R}}$ the modular group of φ_n and by φ the normal state on $\mathcal{A}$ which is the ultraproduct of the states φ_n. $p \in \mathcal{A}$ denote the support of φ. Consider $\widetilde{\mathcal{N}}$ a w^*-dense $*$-subalgebra of $\mathcal{N}$ and a $*$-homomorphism Φ*

$$\Phi \ : \ \widetilde{\mathcal{N}} \subset \mathcal{N} \longrightarrow \mathcal{A} = \prod_{n,\mathcal{U}} \mathcal{A}_n$$

Assume Φ satisfies:

1. *Φ is state preserving: for all $x \in \widetilde{\mathcal{N}}$ we have*

$$\varphi(\Phi(x)) = \psi(x)$$

2. *For all $(x, y) \in \Phi(\widetilde{\mathcal{N}})^2$*

$$\varphi(xy) = \varphi(xpy).$$

   (Or one of the technical conditions of Lemma 4.1.)

3. *For all $t \in \mathbb{R}$ and for all $y = (y_n)^\bullet_{n\in\mathbb{N}} \in \Phi(\widetilde{\mathcal{N}})$,*

$$p(\sigma_t^n(y_n))^\bullet_{n\in\mathbb{N}}p \ (= \sigma_t(pyp)) \in p\mathcal{B}p$$

   where $\mathcal{B}$ is the w^*-closure of $\Phi(\widetilde{\mathcal{N}})$ in $\mathcal{A}$.

Then $\Theta = p\Phi p \ : \ \widetilde{\mathcal{N}} \longrightarrow p\mathcal{A}p$ is a state preserving $*$-homomorphism which can be extended into a normal isomorphism (still denoted by Θ) between $\mathcal{N}$ and its image $\Theta(\mathcal{N}) = p\mathcal{B}p$. Moreover there exists a (normal) state preserving conditional expectation from $\mathcal{A}$ onto $\Theta(\mathcal{N})$.

Remarks.

• Condition 2. is in fact necessary for Θ being a $*$-homomorphism (by Lemma 4.1), and condition 3. is necessary for the existence of a state preserving conditional expectation onto $\Theta(\mathcal{N})$ (by [Tak]).

• Let us denote by $(\sigma_t^\psi)_{t\in\mathbb{R}}$ the modular group of $*$-automorphisms of ψ. Provided that $(\sigma_t^\psi)_{t\in\mathbb{R}}$ maps $\widetilde{\mathcal{N}}$ into itself, we can replace condition 2. of the previous Theorem by the following intertwining condition: For all $t \in \mathbb{R}$ and for all $x \in \widetilde{\mathcal{N}}$ we have

$$p(\sigma_t^n(y_n))^\bullet_{n\in\mathbb{N}}p \; (= \sigma_t(p\Phi(x)p)) = p\Phi(\sigma_t^\psi(x))p$$

where $\Phi(x) = (y_n)^\bullet_{n\in\mathbb{N}}$. Moreover, notice that if the conclusion of the Theorem is true, then this condition must be fulfilled for all $t \in \mathbb{R}$ and for all $x \in \mathcal{N}$ (see [Tak2] page 95).

Corollary 4.4 *Under the assumptions of the previous Theorem, $\mathcal{N}$ is QWEP provided that each of the $\mathcal{A}_n$ is QWEP.*

Proof. This is a consequence of Kirchberg's results (see [Kir, Oz]). First, $\prod\limits_{n\in\mathbb{N}} \mathcal{A}_n$ is QWEP as a product of QWEP C^*-algebras ([Oz] Proposition 4.1 (i)). Since $\tilde{\mathcal{A}}$ is a quotient of a QWEP C^*-algebra, it is also QWEP. It follows that $\mathcal{A}$ which is the w^*-closure of $\tilde{\mathcal{A}}$ in $B(H)$ is QWEP (by [Oz] Proposition 4.1 (iii)). Since there is a conditional expectation from $\mathcal{A}$ onto $p\mathcal{A}p$, $p\mathcal{A}p$ is QWEP (see [Kir]). Finally, by Theorem 4.3, $\mathcal{N}$ is isomorphic to a subalgebra of $p\mathcal{A}p$ which is the image of a (state preserving) conditional expectation, thus $\mathcal{N}$ inherits the QWEP property. $\square$

5 The finite dimensional case

In this section we show that $\Gamma_q(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}})$ is QWEP when $H_\mathbb{R}$ is finite dimensional. For notational purpose, it will be more convenient to deal with $\dim(H_\mathbb{R})$ even. This is not relevant in our context (see the remark after Theorem 5.8). We put $\dim(H_\mathbb{R}) = 2k$. Notice that $\Gamma_q(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}})$ only depends on the spectrum of the operator A. The spectrum of A is given by the set $\{\lambda_1, \dots, \lambda_k\} \cup \{\lambda_1^{-1}, \dots, \lambda_k^{-1}\}$ where for all $j \in \{1, \dots, k\}$, $\lambda_j \geqslant 1$. As in subsection 2.2 , we use the notation $\mu_j = \lambda_j^{\frac{1}{4}}$.

5.1 Twisted Baby Fock

We start by adapting Biane's model to our situation. Let us denote by I the set $\{-k, \dots, -1\} \cup \{1, \dots, k\}$. As in subsection 2.4, we give us a function ϵ on $I \times I$ into $\{-1, 1\}$ and we consider the associated complex $*$-algebra $\mathcal{A}(I, \epsilon)$. By analogy with (3), for all $j \in \{1, \dots, k\}$ we define the following generalized semi-circular variables acting on $L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon)$:

$$\gamma_i = \mu_i^{-1}\beta_i^* + \mu_i\beta_{-i} \quad \text{and} \quad \delta_i = \mu_i\alpha_i^* + \mu_i^{-1}\alpha_{-i}$$

We denote by Γ (respectively Γ_r) the von Neumann algebra generated in $B(L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon))$ by the γ_i (respectively δ_i). Γ_r is the natural candidate for the commutant of Γ in $B(L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon))$. We need to show that the vector 1 is cyclic and separating for Γ. To do so we must assume that ϵ satisfies the following additional condition:

$$\text{For all} \quad (i, j) \in I^2, \quad \epsilon(i, j) = \epsilon(|i|, |j|) \tag{17}$$

This condition is in fact a necessary condition for $\Gamma_r \subset \Gamma'$ and for condition 1.(a) of Lemma 5.2 below.

Lemma 5.1 *Under condition (17) the following relation holds:*

$$\textit{For all} \quad i \in I, \quad \alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i} = \beta_i^*\alpha_i + \beta_{-i}\alpha_{-i}^*$$

Proof. Let $i \in I$ and $A \subset I$. We have

$$(\alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i})(x_A) = \begin{cases} x_{-i}x_Ax_{-i} & \text{if} \quad i \in A \quad \text{and} \quad -i \in A \\ 0 & \text{if} \quad i \in A \quad \text{and} \quad -i \notin A \\ x_ix_Ax_i + x_{-i}x_Ax_{-i} & \text{if} \quad i \notin A \quad \text{and} \quad -i \in A \\ x_ix_Ax_i & \text{if} \quad i \notin A \quad \text{and} \quad -i \notin A \end{cases}$$

and

$$(\beta_i^*\alpha_i + \beta_{-i}\alpha_{-i}^*)(x_A) = \begin{cases} x_ix_Ax_i & \text{if} \quad i \in A \quad \text{and} \quad -i \in A \\ x_ix_Ax_i + x_{-i}x_Ax_{-i} & \text{if} \quad i \in A \quad \text{and} \quad -i \notin A \\ 0 & \text{if} \quad i \notin A \quad \text{and} \quad -i \in A \\ x_{-i}x_Ax_{-i} & \text{if} \quad i \notin A \quad \text{and} \quad -i \notin A \end{cases}$$

Thus, we need to study the following cases. Assume that $A = \{i_1, \dots, i_p\}$ where $i_1 < \cdots < i_p$.

1. If i and $-i$ belong to A then there exists $(l, m) \in \{1, \dots, p\}$, $l < m$, such that $i_l = -i$ and $i_m = i$. Applying successively relations (4) and (17), we get:

$$\begin{aligned} x_{-i}x_Ax_{-i} &= \left(\prod_{q=1}^{l-1} \epsilon(i_q, -i)\right) x_{i_1} \dots x_{i_{l-1}} x_{i_{l+1}} \dots x_{i_p} x_{-i} \\ &= \left(\prod_{q=1}^{l-1} \epsilon(i_q, -i)\right) \left(\prod_{q=l+1}^{p} \epsilon(i_q, -i)\right) x_A = -\left(\prod_{q=1}^{p} \epsilon(i_q, -i)\right) x_A \\ &= -\left(\prod_{q=1}^{p} \epsilon(i_q, i)\right) x_A = x_ix_Ax_i \end{aligned}$$

2. If i and $-i$ do not belong to A, we can check in a similar way that:

$$\begin{aligned} x_{-i}x_Ax_{-i} = \left(\prod_{q=1}^{p} \epsilon(i_q, -i)\right) x_A = \left(\prod_{q=1}^{p} \epsilon(i_q, i)\right) x_A \\ = x_ix_Ax_i \end{aligned}$$

3. If $i \in A$ and $-i \notin A$, then there exists $l \in \{1, \dots, p\}$ such that $i_l = i$. We have:

$$\begin{aligned} x_ix_Ax_i &= \left(\prod_{q=1}^{l-1} \epsilon(i_q, i)\right) x_{i_1} \dots x_{i_{l-1}} x_{i_{l+1}} \dots x_{i_p} x_i \\ &= \left(\prod_{q=1}^{l-1} \epsilon(i_q, i)\right) \left(\prod_{q=l+1}^{p} \epsilon(i_q, i)\right) x_A = -\left(\prod_{q=1}^{p} \epsilon(i_q, i)\right) x_A \\ &= -\left(\prod_{q=1}^{p} \epsilon(i_q, -i)\right) x_A = -x_{-i}x_Ax_{-i} \end{aligned}$$

This finishes the proof.

$\square$

Lemma 5.2 *By construction we have:*

1. *For all $(i,j) \in \{1,\dots,k\}^2$, $i \neq j$, the following mixed commutation and anti-commutation relations hold:*

   (a) $\gamma_i\gamma_j - \epsilon(i,j)\gamma_j\gamma_i = 0$

   (b) $\gamma_i^*\gamma_j - \epsilon(i,j)\gamma_j\gamma_i^* = 0$

   (c) $(\gamma_i^*)^2 = \gamma_i^2 = 0$

   (d) $\gamma_i^*\gamma_i + \gamma_i\gamma_i^* = (\mu_i^2 + \mu_i^{-2})Id.$

2. *Same relations as in 1. for the operators δ_i.*

3. $\Gamma_r \subset \Gamma'$.

4. *The vector* 1 *is cyclic and separating for both* Γ *and* Γ_r.

5. $\Gamma \subset B(L^2(\mathcal{A}(I,\epsilon),\varphi^\epsilon))$ *is the (faithful) G.N.S representation of* $(\Gamma, \varphi^\epsilon)$.

Proof. 1.(a) Thanks to 2. of Lemma 2.3 and (17) we get:

$$\begin{aligned}
\gamma_i\gamma_j &= \mu_i^{-1}\mu_j^{-1}\beta_i^*\beta_j^* + \mu_i\mu_j\beta_{-i}\beta_{-j} + \mu_i^{-1}\mu_j\beta_i^*\beta_{-j} + \mu_i\mu_j^{-1}\beta_{-i}\beta_j^* \\
&= \epsilon(i,j)\mu_i^{-1}\mu_j^{-1}\beta_j^*\beta_i^* + \epsilon(-i,-j)\mu_i\mu_j\beta_{-j}\beta_{-i} + \epsilon(i,-j)\mu_i^{-1}\mu_j\beta_{-j}\beta_i^* \\
&\qquad + \epsilon(-i,j)\mu_i\mu_j^{-1}\beta_j^*\beta_{-i} \\
&= \epsilon(i,j)\left(\mu_i^{-1}\mu_j^{-1}\beta_j^*\beta_i^* + \mu_i\mu_j\beta_{-j}\beta_{-i} + \mu_i^{-1}\mu_j\beta_{-j}\beta_i^* + \mu_i\mu_j^{-1}\beta_j^*\beta_{-i}\right) \\
&= \epsilon(i,j)\gamma_j\gamma_i
\end{aligned}$$

1.(b) Is analogous to (a) and is left to the reader.
1.(c) Using 1. and 2. of Lemma 2.3, and $\epsilon(i,-i) = \epsilon(i,i) = -1$ we get:

$$\begin{aligned}
\gamma_i^2 &= \mu_i^{-2}(\beta_i^*)^2 + \mu_i^2\beta_{-i}^2 + \beta_i^*\beta_{-i} + \beta_{-i}\beta_i^* \\
&= \epsilon(i,-i)\beta_{-i}\beta_i^* + \beta_{-i}\beta_i^* = 0
\end{aligned}$$

1.(d) Using similar arguments, we compute:

$$\begin{aligned}
\gamma_i^*\gamma_i + \gamma_i\gamma_i^* &= \mu_i^{-2}(\beta_i\beta_i^* + \beta_i^*\beta_i) + \mu_i^2(\beta_{-i}^*\beta_{-i} + \beta_{-i}\beta_{-i}^*) + \beta_i\beta_{-i} + \beta_{-i}\beta_i \\
&\qquad + \beta_{-i}^*\beta_i^* + \beta_i^*\beta_{-i}^* \\
&= (\mu_i^{-2} + \mu_i^2)Id + (\epsilon(i,-i)+1)(\beta_i\beta_{-i} + \beta_{-i}^*\beta_i^*) = (\mu_i^{-2} + \mu_i^2)Id
\end{aligned}$$

2. Is now clear from the proof of 1. since the relations for the α_i's are the same as the ones for the β_i's.
3. It suffices to show that for all $(i,j) \in \{1,\dots,k\}^2$ we have $\gamma_i\delta_j = \delta_j\gamma_i$ and $\gamma_i\delta_j^* = \delta_j^*\gamma_i$.
If $i \neq j$ then from 5. of Lemma 2.3 it is clear that $\gamma_i\delta_j = \delta_j\gamma_i$ and $\gamma_i\delta_j^* = \delta_j^*\gamma_i$.
If $i = j$ then using 4. and 5. of Lemma 2.3 and Lemma 5.1 we obtain the desired result as follows:

$$\begin{aligned}
\gamma_i\delta_i &= \beta_i^*\alpha_i^* + \beta_{-i}\alpha_{-i} + \mu_i^{-2}\beta_i^*\alpha_{-i} + \mu_i^2\beta_{-i}\alpha_i^* = \mu_i^{-2}\beta_i^*\alpha_{-i} + \mu_i^2\beta_{-i}\alpha_i^* \\
&= \mu_i^{-2}\alpha_{-i}\beta_i^* + \mu_i^2\alpha_i^*\beta_{-i} = \delta_i\gamma_i
\end{aligned}$$

and

$$\begin{aligned}\gamma_i\delta_i^* &= \beta_i^*\alpha_i + \beta_{-i}\alpha_{-i}^* + \mu_i^{-2}\beta_i^*\alpha_{-i}^* + \mu_i^2\beta_{-i}\alpha_i\\ &= \alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i} + \mu_i^{-2}\beta_i^*\alpha_{-i} + \mu_i^2\beta_{-i}\alpha_i^*\\ &= \alpha_i\beta_i^* + \alpha_{-i}^*\beta_{-i} + \mu_i^{-2}\alpha_{-i}\beta_i^* + \mu_i^2\alpha_i^*\beta_{-i} = \delta_i^*\gamma_i\end{aligned}$$

4. It suffices to prove that for any $A \subset I$ we have $x_A \in \Gamma 1 \cap \Gamma_r 1$. Let $A \subset I$ and $(\chi_i)_{i\in I} \in \{0,1\}^I$ such that $\chi_i = 1$ if and only if $i \in A$. Then

$$\begin{aligned}x_A &= x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k}\\ &= (\mu_k^{-1}\gamma_k^*)^{\chi_{-k}} \dots (\mu_1^{-1}\gamma_1^*)^{\chi_{-1}} (\mu_1\gamma_1)^{\chi_1} \dots (\mu_k\gamma_k)^{\chi_k} 1\\ &= \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}} \gamma_k^{-\chi_{-k}} \dots \gamma_1^{-\chi_{-1}} \gamma_1^{\chi_1} \dots \gamma_k^{\chi_k} 1\end{aligned}$$

where by convention $\gamma_i^{-1} = \gamma_i^*$.
The same computation is valid for Γ_r and we obtain:

$$x_A = \mu_1^{\chi_{-1}-\chi_1} \dots \mu_k^{\chi_{-k}-\chi_k} \delta_k^{\chi_k} \dots \delta_1^{\chi_1} \delta_1^{-\chi_{-1}} \dots \delta_k^{-\chi_{-k}} 1$$

It follows that the vector 1 is cyclic for both Γ and Γ_r. Since $\Gamma_r \subset \Gamma'$ then 1 is also cyclic for Γ' and thus separating for Γ. The same argument applies to Γ_r and thus 1 is also a cyclic and separating vector for Γ_r.
5. This is clear from the just proved assertion and the fact that the state φ^ϵ is equal to the vector state associated to the vector 1. ☐

By the Lemma just proved, we are in a situation where we can apply Tomita-Takesaki theory. As usual we denote by S the involution on $L^2(\mathcal{A}(I,\epsilon), \varphi^\epsilon)$ defined by: $S(\gamma 1) = \gamma^* 1$ for all $\gamma \in \Gamma$. Δ will denote the modular operator and J the modular conjugation. Recall that $S = J\Delta^{\frac{1}{2}}$ is the polar decomposition of the antilinear operator S (which is here bounded since we are in a finite dimensional framework). We also denote by $(\sigma_t)_{t\in\mathbb{R}}$ the modular group of automorphisms of Γ associated to φ. Recall that for all $\gamma \in \Gamma$ and all $t \in \mathbb{R}$ we have $\sigma_t(\gamma) = \Delta^{it}\gamma\Delta^{-it}$.

Notation: In the following, for $A \subset I$ we denote by $(\chi_i)_{i\in I}$ the characteristic function of the set A : $\chi_i = 1$ if $i \in A$ and $\chi_i = 0$ if $i \notin A$. (We will not keep track of the dependance in A unless there could be some confusion.)

Proposition 5.3 *The modular operators and the modular group of $(\Gamma, \varphi^\epsilon)$ are determined by:*

1. *J is the antilinear operator given by: for all $A \subset I$,*

$$J(x_A) = J(x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k}) = x_{-k}^{\chi_k} \dots x_{-1}^{\chi_1} x_1^{\chi_{-1}} \dots x_k^{\chi_{-k}}$$

2. *Δ is the diagonal and positive operator given by: for all $A \subset I$,*

$$\Delta(x_A) = \Delta(x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k}) = \lambda_k^{(\chi_k-\chi_{-k})} \dots \lambda_1^{(\chi_1-\chi_{-1})} x_A$$

3. *For all $j \in \{1\dots,k\}$, γ_j is entire for $(\sigma_t)_t$ and satisfies $\sigma_z(\gamma_j) = \lambda_j^{iz}\gamma_j$ for all $z \in \mathbb{C}$.*

Proof. Let $A \subset I$. We have

$$x_A = x_{-k}^{\chi_{-k}} \dots x_{-1}^{\chi_{-1}} x_1^{\chi_1} \dots x_k^{\chi_k} = \mu_1^{\chi_1 - \chi_{-1}} \dots \mu_k^{\chi_k - \chi_{-k}} \gamma_k^{-\chi_{-k}} \dots \gamma_1^{-\chi_{-1}} \gamma_1^{\chi_1} \dots \gamma_k^{\chi_k} 1$$

Thus,

$$\begin{aligned} S(x_A) &= \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k - \chi_{-k}} (\gamma_k^{-\chi_{-k}} \dots \gamma_1^{-\chi_{-1}} \gamma_1^{\chi_1} \dots \gamma_k^{\chi_k})^* 1 \\ &= \mu_1^{\chi_1-\chi_{-1}} \dots \mu_k^{\chi_k-\chi_{-k}} \gamma_k^{-\chi_k} \dots \gamma_1^{-\chi_1} \gamma_1^{\chi_{-1}} \dots \gamma_k^{\chi_{-k}} 1 \\ &= \mu_1^{2(\chi_1-\chi_{-1})} \dots \mu_k^{2(\chi_k-\chi_{-k})} x_{-k}^{\chi_k} \dots x_{-1}^{\chi_1} x_1^{\chi_{-1}} \dots x_k^{\chi_{-k}} \end{aligned}$$

By uniqueness of the polar decomposition, we obtain the stated result. Let $j \in \{1 \dots k\}$ and $t \in \mathbb{R}$ we have:

$$\begin{aligned} \sigma_t(\gamma_j) 1 &= \Delta^{it} \gamma_j \Delta^{-it} 1 = \Delta^{it} \gamma_j 1 = \mu_j^{-1} \Delta^{it} x_j = \mu_j^{-1} \mu_j^{4it} x_j \\ &= \mu_j^{4it} \gamma_j 1 \end{aligned}$$

It follows, since 1 is separating for Γ, that $\sigma_t(\gamma_j) = \mu_j^{4it} \gamma_j$. □

Remarks.

• We have $\Gamma' = \Gamma_r$. Indeed we have already proved the inclusion $\Gamma_r \subset \Gamma'$ in Lemma 5.2 . For the reverse inclusion we can use Tomita-Takesaki theory which ensures that $\Gamma' = J \Gamma J$. But for all $j \in I$ it is easy to see that $J \beta_j J = \alpha_{-j}$. It follows that for all $j \in \{1, \dots, k\}$ we have $J \gamma_j J = \delta_j^*$. Thus $\Gamma' \subset \Gamma_r$. The equality $\Gamma' = \Gamma_r$ can also be seen as a consequence of a general fact in Tomita-Takesaki theory: it suffices to remark that Γ_r is the right Hilbertian algebra associated to Γ in its GNS representation.

• The previous construction can be performed for an infinite set of the form $J \times \{-1, 1\}$ given with a family of eigenvalues $(\mu_j)_{j \in J} \in [1, +\infty[^J$ and a sign function ϵ satisfying

$$\epsilon((j,i),(j',i')) = \epsilon((j,1),(j',1)) \quad \text{for all} \quad ((j,i),(j',i')) \in (J \times \{-1,1\})^2.$$

5.2 Central limit approximation of q-Gaussians

In this section we use the twisted Baby Fock construction to obtain an asymptotic random matrix model for the q-Gaussian variables, via Speicher's central limit Theorem. Let us first check the independence condition:

Lemma 5.4 *For all $j \in \{1, \dots, k\}$ let us denote by $\mathcal{A}_j$ the C^*-subalgebra of $B(L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon))$ generated by the operators β_j and β_{-j}. Then the family $(\mathcal{A}_j)_{1 \leqslant j \leqslant k}$ is independent in $B(L^2(\mathcal{A}(I, \epsilon), \varphi^\epsilon))$. In particular, the family $(\gamma_j)_{1 \leqslant j \leqslant k}$ is independent.*

Proof. The proof proceeds by induction. Changing notation, it suffices to show that

$$\varphi^\epsilon(a_1 \dots a_{r+1}) = \varphi^\epsilon(a_1 \dots a_r) \varphi^\epsilon(a_{r+1})$$

where $a_l \in \mathcal{A}_l$ for all $l \in \{1, \dots, r+1\}$. Since a_{r+1} is a certain non-commutative polynomial in the variables β_{r+1}, β_{r+1}^*, $\beta_{-(r+1)}$, and $\beta_{-(r+1)}^*$, it is clear that there exists $\nu \in \mathrm{Span}\{x_{r+1},\ x_{-(r+1)},\ x_{-(r+1)} x_{r+1}\}$ such that

$$a_{r+1} 1 = \langle 1, a_{r+1} 1 \rangle 1 + \nu$$

It is easy to see that $a_r^* \dots a_1^* 1 \in \mathrm{Span}\,\{x_B,\ B \subset \{-r, \dots, -1\} \cup \{1, \dots, r\}\}$, which is orthogonal to $\mathrm{Span}\{x_{r+1},\ x_{-(r+1)},\ x_{-(r+1)}x_{r+1}\}$. We compute:

$$\begin{aligned}\varphi^\epsilon(a_1 \dots a_{r+1}) &= \langle 1, a_1 \dots a_r a_{r+1} 1\rangle = \langle a_r^* \dots a_1^* 1, a_{r+1} 1\rangle \\ &= \langle a_r^* \dots a_1^* 1, 1\rangle\langle 1, a_{r+1} 1\rangle + \langle a_r^* \dots a_1^* 1, \nu\rangle = \langle 1, a_1 \dots a_r 1\rangle\langle 1, a_{r+1} 1\rangle \\ &= \varphi^\epsilon(a_1 \dots a_r)\varphi^\epsilon(a_{r+1})\end{aligned}$$

□

Remark. It is clear that one can prove, in the same way, that the C^*-algebras generated by the β_j are independent (this is Proposition 3 in [Bi]).

Let $q \in (-1, 1)$. Let us choose a family of random variables $(\epsilon(i,j))_{(i,j)\in\mathbb{N}_*^2, i\neq j}$ as in Lemma 2.6, and set $\epsilon(i,i) = -1$ for all $i \in \mathbb{N}_*$. As in section 2.5.1, for all $n \in \mathbb{N}_*$ we will consider the complex $*$-algebra $\mathcal{A}(I_n, \epsilon_n)$ where

$$I_n = \{1, \dots, n\} \times \big(\{-k, \dots, -1\} \cup \{1, \dots, k\}\big)$$

and

$$\epsilon_n((i,j),(i',j')) = \epsilon(i,i') \quad \text{for all} \quad ((i,j),(i',j')) \in I_n^2.$$

Notice that the analogue of condition (17) is automatically satisfied. Indeed, we have:

$$\epsilon_n((i,j),(i',j')) = \epsilon_n((i,|j|),(i',|j'|)) \quad \text{for all} \quad ((i,j),(i',j')) \in I_n^2.$$

Let us remind that $\mathcal{A}(I_n, \epsilon_n)$ is the unital free complex algebra with generators $(x_{i,j})_{(i,j)\in I_n}$ quotiented by the relations,

$$x_{i,j}x_{i',j'} - \epsilon(i,i')x_{i',j'}x_{i,j} = 2\delta_{(i,j),(i',j')}$$

and with involution given by $x_{i,j}^* = x_{i,j}$. For all $(i,j) \in \{1, \dots, n\} \times \{1, \dots, k\}$ let $\gamma_{i,j}$ be the "twisted semi-circular variable" associated to μ_j

$$\gamma_{i,j} = \mu_j^{-1}\beta_{i,j}^* + \mu_j\beta_{i,j}$$

We denote by $\Gamma_n \subset B(L^2(\mathcal{A}(I_n, \epsilon_n), \varphi^{\epsilon_n}))$ the von-Neumann algebra generated by the $\gamma_{i,j}$ for $(i,j) \in \{1, \dots, n\} \times \{1, \dots, k\}$. Observe that all our notations are consistent since $(\Gamma_n, \varphi^{\epsilon_n})$ is naturally embedded in $(\Gamma_{n+1}, \varphi^{\epsilon_{n+1}})$ (see the remarks following Lemma (2.3)). In fact all these algebras $(\Gamma_n, \varphi^{\epsilon_n})$ can be embedded in the bigger von Neumann algebra $(\Gamma, \varphi^{\overline{\epsilon}})$ which is the Baby Fock construction associated to the infinite set $\overline{I}$ and the sign function $\overline{\epsilon}$ given by

$$\overline{I} = \mathbb{N}_* \times \big(\{-k, \dots, -1\} \cup \{1, \dots, k\}\big)$$

and

$$\overline{\epsilon}((i,j),(i',j')) = \epsilon(i,i') \quad \text{for all} \quad ((i,j),(i',j')) \in \overline{I}^2.$$

Let us denote by $s_{n,j}$ the following sum:

$$s_{n,j} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\gamma_{i,j}$$

We now check the hypothesis of Theorem 2.5 for the family $(\gamma_{i,j})_{(i,j)\in\mathbb{N}_*\times\{1,\dots,k\}} \subset (\Gamma, \varphi^{\overline{\epsilon}})$.

1. The family is independent by Lemma 5.4.

2. It is clear that for all (i,j) we have $\varphi^{\overline{\epsilon}}(\gamma_{i,j}) = 0$.

3. Let $(j(1), j(2)) \in \{1, \ldots, k\}$ and $i \in \mathbb{N}_*$. We compute and identify the covariance thanks to Lemma 2.2:

$$\begin{aligned}\varphi^{\overline{\epsilon}}(\gamma_{i,j(1)}^{k(1)}\gamma_{i,j(2)}^{k(2)}) &= \langle \gamma_{i,j(1)}^{-k(1)} 1, \gamma_{i,j(2)}^{k(2)} 1 \rangle = \langle \mu_{j(1)}^{k(1)} x_{-k(1)i,-k(1)j(1)}, \mu_{j(2)}^{-k(2)} x_{k(2)i,k(2)j(2)} \rangle \\ &= \mu_{j(1)}^{2k(1)} \delta_{k(2),-k(1)} \delta_{j(1),j(2)} = \varphi(c_{j(1)}^{k(1)} c_{j(2)}^{k(2)})\end{aligned}$$

4. It is easily seen that $\varphi^{\overline{\epsilon}}(\gamma_{i,j}^{k(1)} \ldots \gamma_{i,j}^{k(w)})$ is independent of $i \in \mathbb{N}_*$.

5. This is a consequence of Lemma 5.2.

6. This follows from Lemma 2.6 almost surely.

Thus, by Theorem 2.5, we have, almost surely, for all $p \in \mathbb{N}_*$, $(k(1), \ldots, k(p)) \in \{-1,1\}^p$ and all $(j(1), \ldots, j(p)) \in \{1, \ldots, k\}^p$:

$$\lim_{n \to +\infty} \varphi^{\overline{\epsilon}}(s_{n,j(1)}^{k(1)} \ldots s_{n,j(p)}^{k(p)}) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \displaystyle\sum_{\substack{\mathcal{V} \in \mathcal{P}_2(1,\ldots,2r) \\ \mathcal{V} = \{(s_l,t_l)_{l=1}^{l=r}\}}} q^{i(\mathcal{V})} \prod_{l=1}^{r} \varphi(c_{j(s_l)}^{k(s_l)} c_{j(t_l)}^{k(t_l)}) & \text{if } p = 2r \end{cases}$$

By Lemma 2.2 we see that all $*$-moments of the family $(s_{n,j})_{j \in \{1,\ldots,k\}}$ converge when n goes to infinity to the corresponding $*$-moments of the family $(c_j)_{j \in \{1,\ldots,k\}}$:

Proposition 5.5 *For all $p \in \mathbb{N}_*$, $(j(1), \ldots, j(p)) \in \{1, \ldots, k\}^p$ and for all $(k(1), \ldots, k(p)) \in \{-1,1\}^p$ we have:*

$$\lim_{n \to +\infty} \varphi^{\overline{\epsilon}}(s_{n,j(1)}^{k(1)} \ldots s_{n,j(p)}^{k(p)}) = \varphi(c_{j(1)}^{k(1)} \ldots c_{j(p)}^{k(p)}) \quad \textit{almost surely} \tag{18}$$

Remark. It is possible (and maybe easier) to apply directly Speicher's Theorem to the independent family $(\beta_{i,j})_{(i,j) \in \overline{I}^2}$. Then, it suffices to follow the analogies between the Baby Fock and the q-Fock frameworks to deduce the previous Proposition.

5.3 $\Gamma_q(H_{\mathbb{R}}, (U_t)_{t \in \mathbb{R}})$ is QWEP

For all $j \in \{1, \ldots, k\}$ let us denote by $g_{n,j} = \mathrm{Re}\,(s_{n,j})$ and $g_{n,-j} = \mathrm{Im}\,(s_{n,j})$. By (18) we have that for all monomials P in $2k$ noncommuting variables:

$$\lim_{n \to +\infty} \varphi^{\overline{\epsilon}}(P(g_{n,-k}, \ldots, g_{n,k})) = \varphi(P(G(f_{-k}), \ldots, G(f_k))) \quad \text{almost surely} \tag{19}$$

Since the set of all non-commutative monomials is countable, we can find a choice of signs ϵ such that (19) is true for all P. In the sequel we fix such an ϵ and forget about the dependance on ϵ.

Lemma 5.6 *For all polynomials P in $2k$ noncommuting variables we have:*

$$\lim_{n\to+\infty} \varphi(P(g_{n,-k},\dots,g_{n,k})) = \varphi(P(G(f_{-k}),\dots,G(f_k))) \tag{20}$$

We are now ready to construct an embedding of $\Gamma_q(H_\mathbb{R},U_t)$ into an ultraproduct of the finite dimensional von Neumann algebras Γ_n. To do so we need to have a uniform bound on the operators $g_{n,j}$. Let $C>0$ such that for all $j\in I$, $\|G(f_j)\|<C$, as in the tracial case, we replace the $g_{n,j}$ by the their truncations $\tilde{g}_{n,j}=\chi_{]-C,C[}(g_{n,j})g_{n,j}$. The following is the analogue of Lemma 3.1:

Lemma 5.7 *For all polynomials P in $2k$ noncommuting variables we have:*

$$\lim_{n\to+\infty} \varphi(P(\tilde{g}_{n,-k},\dots,\tilde{g}_{n,k})) = \varphi(P(G(f_{-k}),\dots,G(f_k))) \tag{21}$$

Remark. For all $n\in\mathbb{N}_*$ and all $j\in I$ the element $g_{n,j}$ is entire for the modular group (this is always the case in a finite dimensional framework). By (3) of proposition 5.3, we have for all $j\in\{1,\dots,k\}$

$$\sigma_z(s_{n,j}) = \lambda_j^{iz}s_{n,j} \quad \text{for all } z\in\mathbb{C}$$

Thus for all $z\in\mathbb{C}$,

$$\sigma_z(g_{n,j}) = \begin{cases} \cos(z\ln(\lambda_j))g_{n,j} - \sin(z\ln(\lambda_j))g_{n,-j} & \text{for all } j\in\{1,\dots,k\} \\ \sin(z\ln(\lambda_{-j}))g_{n,-j} + \cos(z\ln(\lambda_{-j}))g_{n,j} & \text{for all } j\in\{-1,\dots,-k\} \end{cases} \tag{22}$$

Proof of Lemma 5.7. It suffices to show that for all $(j(1),\dots,j(p))\in I^p$ we have

$$\lim_{n\to+\infty} \varphi(\tilde{g}_{n,j(1)}\dots\tilde{g}_{n,j(p)}) = \varphi(G(f_{j(1)})\dots G(f_{j(p)}))$$

By (20) it is sufficient to prove that

$$\lim_{n\to+\infty} |\varphi(g_{n,j(1)}\dots g_{n,j(p)}) - \varphi(\tilde{g}_{n,j(1)}\dots\tilde{g}_{n,j(p)})| = 0$$

Using multi-linearity we can write

$$\begin{aligned} &|\varphi(g_{n,j(1)}\dots g_{n,j(p)}) - \varphi(\tilde{g}_{n,j(1)}\dots\tilde{g}_{n,j(p)})| \\ &\qquad = |\sum_{l=1}^{p} \varphi[\tilde{g}_{n,j(1)}\dots\tilde{g}_{n,j(l-1)}(g_{n,j(l)}-\tilde{g}_{n,j(l)})g_{n,j(l+1)}\dots g_{n,j(p)}]| \\ &\qquad \leqslant \sum_{l=1}^{p} |\varphi[\tilde{g}_{n,j(1)}\dots\tilde{g}_{n,j(l-1)}(g_{n,j(l)}-\tilde{g}_{n,j(l)})g_{n,j(l+1)}\dots g_{n,j(p)}]| \end{aligned}$$

Fix $l\in\{1,\dots,p\}$, using the modular group we have:

$$\begin{aligned} &|\varphi[\tilde{g}_{n,j(1)}\dots\tilde{g}_{n,j(l-1)}(g_{n,j(l)}-\tilde{g}_{n,j(l)})g_{n,j(l+1)}\dots g_{n,j(p)}]| \\ &\qquad = |\varphi[\sigma_i(g_{n,j(l+1)}\dots g_{n,j(p)})\tilde{g}_{n,j(1)}\dots\tilde{g}_{n,j(l-1)}(g_{n,j(l)}-\tilde{g}_{n,j(l)})]| \end{aligned}$$

Estimating by Cauchy-Schwarz's inequality we obtain:

$$\begin{aligned}
&|\varphi[\sigma_i(g_{n,j(l+1)}\cdots g_{n,j(p)})\tilde{g}_{n,j(1)}\cdots\tilde{g}_{n,j(l-1)}(g_{n,j(l)}-\tilde{g}_{n,j(l)})]| \\
&\qquad\leqslant \varphi[\sigma_i(g_{n,j(l+1)}\cdots g_{n,j(p)})\tilde{g}_{n,j(1)}\cdots\tilde{g}^2_{n,j(l-1)}\cdots\tilde{g}_{n,j(1)}\sigma_{-i}(g_{n,j(p)}\cdots g_{n,j(l+1)})]^{\frac{1}{2}} \\
&\qquad\qquad\times\varphi[(g_{n,j(l)}-\tilde{g}_{n,j(l)})^2]^{\frac{1}{2}} \\
&\qquad\leqslant C^{l-1}\varphi[\sigma_i(g_{n,j(l+1)}\cdots g_{n,j(p)})\sigma_{-i}(g_{n,j(p)}\cdots g_{n,j(l+1)})]^{\frac{1}{2}}\varphi[(g_{n,j(l)}-\tilde{g}_{n,j(l)})^2]^{\frac{1}{2}}
\end{aligned}$$

The conclusion follows from the convergence of this last term to 0. Indeed, by (22) there exists a polynomial in $2k$ non-commutative variables Q, independent on n, such that $Q(g_{n,-k}\dots g_{n,k}) = \sigma_i(g_{n,j(l+1)}\cdots g_{n,j(p)})\sigma_{-i}(g_{n,j(p)}\cdots g_{n,j(l+1)})$. It follows by (20) that

$$\lim_{n\to+\infty}\varphi[\sigma_i(g_{n,j(l+1)}\cdots g_{n,j(p)})\sigma_{-i}(g_{n,j(p)}\cdots g_{n,j(l+1)})] = \varphi(Q(G(f_{-k})\dots G(f_k))).$$

And by Lemma 3.2, $\varphi[(g_{n,j(l)}-\tilde{g}_{n,j(l)})^2]$ converges to 0 when n goes to infinity. $\square$

Let us denote by $\mathcal{P}$ the w^*-dense $*$-subalgebra of $\Gamma_q(H_{\mathbb{R}},U_t)$ generated by the set $\{G(f_j),\ j\in I\}$. We know that $\mathcal{P}$ is isomorphic to the algebra of non-commutative polynomials in $2k$ variables (see the remark after Lemma 3.2). Given $\mathcal{U}$ a non trivial ultrafilter on $\mathbb{N}$, it is thus possible to define the following $*$-homomorphism Φ from $\mathcal{P}$ into the von Neumann ultraproduct $\mathcal{A} = \prod_{n,\mathcal{U}}\Gamma_n$ by:

$$\Phi(P(G(f_{-k}),\dots,G(f_k))) = (P(\tilde{g}_{n,-k},\dots,\tilde{g}_{n,k}))^\bullet_{n\in\mathbb{N}}$$

Indeed the right term is well defined since it is uniformly bounded in norm. Let us check the hypothesis of Theorem 4.3.

1. By Lemma 5.7, Φ is state preserving.

2. It is sufficient to check that condition 2. of Lemma 4.1 is satisfied for every generator $\Phi(G(f_j))$, $j\in I$. Let us fix $j\in I$ and recall that by (22) there are complex numbers ν_j and ω_j (independent of n) such that $\sigma^n_{-i}(g_{n,j}) = \nu_j g_{n,j}+\omega_j g_{n,-j}$. We show that condition 2. of Lemma 4.1 is satisfied for $x=\Phi(G(f_j))$ and $z=\nu_j\Phi(G(f_j))+\omega_j\Phi(G(f_{-j}))$. By w^*-density it is sufficient to consider $y=(y_n)^\bullet_{n\in\mathbb{N}}\in\widetilde{\mathcal{A}}$. Using Lemma 5.7 we have:

$$\begin{aligned}
\varphi(\Phi(G(f_j))y) &= \lim_{n,\mathcal{U}}\varphi_n(\tilde{g}_{n,j}y_n) = \lim_{n,\mathcal{U}}\varphi_n(g_{n,j}y_n) = \lim_{n,\mathcal{U}}\varphi_n(y_n\sigma^n_{-i}(g_{n,j})) \\
&= \lim_{n,\mathcal{U}}\varphi_n(y_n(\nu_j g_{n,j}+\omega_j g_{n,-j})) = \lim_{n,\mathcal{U}}\varphi_n(y_n(\nu_j\tilde{g}_{n,j}+\omega_j\tilde{g}_{n,-j})) \\
&= \varphi(y(\nu_j\Phi(G(f_j))+\omega_j\Phi(G(f_{-j}))))
\end{aligned}$$

3. It suffices to check that the intertwining condition given in the remark of Theorem 4.3 is satisfied for the generators $\Phi(G(f_j)) = (\tilde{g}_{n,j})^\bullet_{n\in N}$:

$$\text{for all } j\in I,\quad \sigma_t(p\Phi(G(f_j))p) = p\Phi(\sigma_t(G(f_j)))p$$

To fix ideas we will suppose that $j\geqslant 0$. Recall that in this case for all $t\in\mathbb{R}$ and for all $n\in\mathbb{N}$, we have

$$\sigma^n_t(g_{n,j}) = \cos(t\ln(\lambda_j))g_{n,j} - \sin(t\ln(\lambda_j))g_{n,-j}.$$

Since the functional calculus commutes with automorphisms, for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$, we have:

$$\sigma_t^n(\tilde{g}_{n,j}) = h(\sigma_t^n(g_{n,j})),$$

where $h(\lambda) = \chi_{]-C,C[}(\lambda)\lambda$, for all $\lambda \in \mathbb{R}$. But by Lemma 5.6,

$$\sigma_t^n(g_{n,j}) = \cos(t\ln(\lambda_j))g_{n,j} - \sin(t\ln(\lambda_j))g_{n,-j}$$

converges in distribution to

$$\cos(t\ln(\lambda_j))G(f_j) - \sin(t\ln(\lambda_j))G(f_{-j}) = \sigma_t(G(f_j))$$

and $\|\sigma_t(G(f_j))\| = \|G(f_j)\| < C$. Thus, by Lemma 3.2, we deduce that $\sigma_t^n(\tilde{g}_{n,j})$ converges in distribution to $\sigma_t(G(f_j))$. On the other hand, by Lemma 5.7,

$$\cos(t\ln(\lambda_j))\tilde{g}_{n,j} - \sin(t\ln(\lambda_j))\tilde{g}_{n,-j}$$

also converges in distribution to

$$\cos(t\ln(\lambda_j))G(f_j) - \sin(t\ln(\lambda_j))G(f_{-j}) = \sigma_t(G(f_j)).$$

Let $y \in \mathcal{A}$, using Raynaud's results we compute:

$$\begin{aligned}\varphi(\sigma_t(p\Phi(G(f_j))p)pyp) &= \varphi((\Delta_n^{it})^\bullet p\Phi(G(f_j))p(\Delta_n^{-it})^\bullet pyp)\\ &= \varphi(p(\Delta_n^{it})^\bullet \Phi(G(f_j))(\Delta_n^{-it})^\bullet pyp)\\ &= \varphi((\Delta_n^{it})^\bullet \Phi(G(f_j))(\Delta_n^{-it})^\bullet py)\end{aligned}$$

Let $z = (z_n)^\bullet_{n\in\mathbb{N}} \in \widetilde{\mathcal{A}}$. By our previous observations, we have:

$$\begin{aligned}\varphi((\Delta_n^{it})^\bullet \Phi(G(f_j))(\Delta_n^{-it})^\bullet z) &= \lim_{n,\mathcal{U}} \varphi_n(\Delta_n^{it}\tilde{g}_{n,j}\Delta_n^{-it}z_n)\\ &= \lim_{n,\mathcal{U}} \varphi_n(\sigma_t^n(\tilde{g}_{n,j})z_n)\\ &= \varphi(\sigma_t(G(f_j))z)\\ &= \lim_{n,\mathcal{U}} \varphi_n((\cos(t\ln(\lambda_j))\tilde{g}_{n,j} - \sin(t\ln(\lambda_j))\tilde{g}_{n,-j})z_n)\\ &= \varphi((\cos(t\ln(\lambda_j))\Phi(G(f_j)) - \sin(t\ln(\lambda_j))\Phi(G(f_{-j})))z)\\ &= \varphi((p\Phi(\sigma_t(G(f_j)))p)zp)\end{aligned}$$

By w^*-density and continuity, we can replace z by py in the previous equality, which gives:

$$\varphi(\sigma_t(p\Phi(G(f_j))p)pyp) = \varphi((p\Phi(\sigma_t(G(f_j)))p)pyp).$$

Thus, taking $y = \sigma_t(p\Phi(G(f_j))p) - p\Phi(\sigma_t(G(f_j)))p$, and by the faithfulness of $\varphi(p\,.\,p)$ we deduce that

$$\sigma_t(p\Phi(G(f_j))p) = p\Phi(\sigma_t(G(f_j)))p \in p\mathrm{Im}(\Phi)p$$

By Theorem 4.3, $\Theta = p\Phi p$ can be extended into a (necessarily injective because state preserving) w^*-continuous $*$-homomorphism from $\Gamma_q(H_\mathbb{R}, U_t)$ into $p\mathcal{A}p$ with a completely complemented image. By its corollary 4.4, since the algebras Γ_n are finite dimensional and a fortiori are QWEP, it follows that $\Gamma_q(H_\mathbb{R}, U_t)$ is QWEP.

Theorem 5.8 *If $H_{\mathbb{R}}$ is a finite dimensional real Hilbert space equipped with a group of orthogonal transformations $(U_t)_{t\in\mathbb{R}}$, then the von Neumann algebra $\Gamma_q(H_{\mathbb{R}}, U_t)$ is QWEP.*

Remark. We have only proved the Theorem for $H_{\mathbb{R}}$ of even dimension over $\mathbb{R}$. We did this only for simplicity of notations. Of course this is not relevant since, if the dimension of $H_{\mathbb{R}}$ is odd, then we just have to consider the real Hilbert space $H_{\mathbb{R}} \oplus \mathbb{R}$ equipped with $(U_t \oplus \mathrm{Id})_{t\in\mathbb{R}}$. $\Gamma_q(H_{\mathbb{R}} \oplus \mathbb{R}, U_t \oplus \mathrm{Id})$ is QWEP by our previous discussion. Let us denote by Q the projection from $H_{\mathbb{R}} \oplus \mathbb{R}$ onto $H_{\mathbb{R}}$, then Q intertwines $(U_t \oplus \mathrm{Id})_{t\in\mathbb{R}}$ and $(U_t)_{t\in\mathbb{R}}$. In this situation we can consider $\Gamma_q(Q)$, the second quantization of Q (see [Hi]), which is a conditional expectation from $\Gamma_q(H_{\mathbb{R}} \oplus \mathbb{R}, U_t \oplus \mathrm{Id})$ onto $\Gamma_q(H_{\mathbb{R}}, U_t)$. Thus $\Gamma_q(H_{\mathbb{R}}, U_t)$ is completely complemented into a QWEP von Neumann algebra, so $\Gamma_q(H_{\mathbb{R}}, U_t)$ is QWEP.

Corollary 5.9 *If $(U_t)_{t\in\mathbb{R}}$ is almost periodic on $H_{\mathbb{R}}$, then $\Gamma_q(H_{\mathbb{R}}, U_t)$ is QWEP.*

Proof. There exist an invariant real Hilbert space H_1, an orthogonal family of invariant 2-dimensional real Hilbert spaces $(H_\alpha)_{\alpha\in A}$ and real eigenvalues $(\lambda_\alpha)_{\alpha\in A}$ greater than 1 such that

$$H_{\mathbb{R}} = H_1 \bigoplus_{\alpha\in A} H_\alpha \quad \text{and} \quad U_{t|H_1} = \mathrm{Id}_{H_1}, \quad U_{t|H_\alpha} = \begin{pmatrix} \cos(t\ln(\lambda_\alpha)) & -\sin(t\ln(\lambda_\alpha)) \\ \sin(t\ln(\lambda_\alpha)) & \cos(t\ln(\lambda_\alpha)) \end{pmatrix}$$

In particular it is possible to find a net $(I_\beta)_{\beta\in B}$ of isometries from finite dimensional subspaces $H_\beta \subset H_{\mathbb{R}}$ into $H_{\mathbb{R}}$, such that for all $\beta \in B$, H_β is stable by $(U_t)_{t\in\mathbb{R}}$ and $\bigcup_{\beta\in B} H_\beta$ is dense in $H_{\mathbb{R}}$. By second quantization, for all $\beta \in B$, there exists an isometric $*$-homomorphism $\Gamma_q(I_\beta)$ from $\Gamma_q(H_\beta, U_{t|H_\beta})$ into $\Gamma_q(H_{\mathbb{R}}, U_t)$, and $\Gamma_q(H_{\mathbb{R}}, U_t)$ is the inductive limit (in the von Neumann algebra's sense) of the algebras $\Gamma_q(H_\beta, U_{t|H_\beta})$. By the previous Theorem, for all $\beta \in B$, $\Gamma_q(H_\beta, U_{t|H_\beta})$ is QWEP, thus $\Gamma_q(H_{\mathbb{R}}, U_t)$ is QWEP, as an inductive limit of QWEP von Neumann algebras. $\square$

6 The general case

We will derive the general case by discretization and an ultraproduct argument similar to that of the previous section.

6.1 Discretization argument

Let $H_{\mathbb{R}}$ be a real Hilbert space and $(U_t)_{t\in\mathbb{R}}$ a strongly continuous group of orthogonal transformations on $H_{\mathbb{R}}$. We denote by $H_{\mathbb{C}}$ the complexification of $H_{\mathbb{R}}$ and by $(U_t)_{t\in\mathbb{R}}$ its extension to a group of unitaries on $H_{\mathbb{C}}$. Let A be the (unbounded) non degenerate positive infinitesimal generator of $(U_t)_{t\in\mathbb{R}}$. For every $n \in \mathbb{N}_*$ let g_n be the bounded Borelian function defined by:

$$g_n = \chi_{]1,1+\frac{1}{2^n}[} + \left(\sum_{k=2^n+1}^{n2^n-1} \frac{k}{2^n} \chi_{[\frac{k}{2^n}, \frac{k+1}{2^n}[} \right) + n\chi_{[n,+\infty[}$$

and

$$f_n(t) = g_n(t)\chi_{\{t>1\}}(t) + \frac{1}{g_n(1/t)} \chi_{\{t<1\}}(t) + \chi_{\{1\}}(t) \quad \text{for all } t \in \mathbb{R}_+$$

It is clear that

$$f_n(t) \nearrow t \quad \text{for all} \quad t \geqslant 1 \qquad \text{and} \qquad f_n(t) = \frac{1}{f_n(1/t)} \quad \text{for all} \quad t \in \mathbb{R}_+^*. \tag{23}$$

For all $n \in \mathbb{N}_*$, let A_n be the invertible positive and bounded operator on $H_\mathbb{C}$ defined by $A_n = f_n(A)$. Denoting by $\mathcal{J}$ the conjugation on $H_\mathbb{C}$, we know, by [Sh], that $\mathcal{J}A = A^{-1}\mathcal{J}$. By the second part of (23), it follows that for all $n \in \mathbb{N}_*$,

$$\mathcal{J}A_n = \mathcal{J}f_n(A) = f_n(A^{-1})\mathcal{J} = f_n(A)^{-1}\mathcal{J} = A_n^{-1}\mathcal{J} \tag{24}$$

Consider the strongly continuous unitary group $(U_t^n)_{t\in\mathbb{R}}$ on $H_\mathbb{C}$ with positive non degenerate and bounded infinitesimal generator given by A_n. By definition, we have $U_t^n = A_n^{it}$. By (24), and since $\mathcal{J}$ is anti-linear, we have for all $n \in \mathbb{N}_*$ and all $t \in \mathbb{R}$:

$$\mathcal{J}U_t^n = \mathcal{J}A_n^{it} = A_n^{it}\mathcal{J} = U_t^n\mathcal{J}$$

It follows that for all $n \in \mathbb{N}_*$ and for all $t \in \mathbb{R}$, $H_\mathbb{R}$ is globally invariant by U_t^n, thus we have

$$U_t^n(H_\mathbb{R}) = H_\mathbb{R}$$

Hence, $(U_t^n)_{t\in\mathbb{R}}$ induces a group of orthogonal transformations on $H_\mathbb{R}$ such that its extension on $H_\mathbb{C}$ has infinitesimal generator given by the discretized operator A_n. In the following we will index by $n \in \mathbb{N}_*$ the objects relative to the discretized von Neumann algebra $\Gamma_n = \Gamma_q\left(H_\mathbb{R}, (U_t^n)_{t\in\mathbb{R}}\right)$. We simply set $\Gamma = \Gamma_q\left(H_\mathbb{R}, (U_t)_{t\in\mathbb{R}}\right)$.

Remark. Notice that $H_\mathbb{C}$ is contractively included in H and all H_n, and that the inclusion $H_\mathbb{R} \subset H$ (respectively $H_\mathbb{R} \subset H_n$) is isometric since $\mathrm{Re}\left(\langle\,.\,,\,.\,\rangle_U\right)_{|H_\mathbb{R}\times H_\mathbb{R}} = \langle\,.\,,\,.\,\rangle_{H_\mathbb{R}}$ (see [Sh]). Moreover for all $n \in \mathbb{N}_*$ the scalar products $\langle\,.\,,\,.\,\rangle_{U^n}$ and $\langle\,.\,,\,.\,\rangle_{H_\mathbb{C}}$ are equivalent on $H_\mathbb{C}$ since A_n is bounded.

Scholie 6.1 *For all ξ and η in $H_\mathbb{C}$ we have:*

$$\lim_{n\to+\infty}\langle\xi,\eta\rangle_{H_n} = \langle\xi,\eta\rangle_H$$

Proof. Let E_A be the spectral resolution of A. Take $\xi \in H_\mathbb{C}$ and denote by μ_ξ the finite positive measure on $\mathbb{R}_+$ given by $\mu_\xi = \langle E_A(.)\xi,\xi\rangle_{H_\mathbb{C}}$. Since for all $\lambda \in \mathbb{R}_+$, $\lim\limits_{n\to+\infty} g\circ f_n(\lambda) = g(\lambda)$, and $g(\lambda) = 2\lambda/(1+\lambda)$ is bounded on $\mathbb{R}_+$, we have by the Lebesgue dominated convergence Theorem:

$$\begin{aligned}\|\xi\|_H^2 &= \langle\frac{2A}{1+A}\xi,\xi\rangle_{H_\mathbb{C}} = \int_{\mathbb{R}_+} g(\lambda)\mathrm{d}\mu_\xi(\lambda)\\ &= \lim_{n\to+\infty}\int_{\mathbb{R}_+} g\circ f_n(\lambda)\mathrm{d}\mu_\xi(\lambda) = \lim_{n\to+\infty}\langle\frac{2A_n}{1+A_n}\xi,\xi\rangle_{H_\mathbb{C}} = \lim_{n\to+\infty}\|\xi\|_{H_n}^2\end{aligned}$$

And we finish the proof by polarization. □

Let E be the vector space given by

$$E = \cup_{k\in\mathbb{N}_*}\chi_{[\frac{1}{k},k]}(A)(H_\mathbb{R})$$

We have

$$\mathcal{J}\chi_{[\frac{1}{k},k]}(A) = \chi_{[\frac{1}{k},k]}(A^{-1})\mathcal{J} = \chi_{[\frac{1}{k},k]}(A)\mathcal{J}$$

thus $E \subset H_{\mathbb{R}}$. Since A is non degenerate,

$$\overline{\cup_{k\in\mathbb{N}_*}\chi_{[\frac{1}{k},k]}(A)(H_{\mathbb{C}})} = \chi_{]0,+\infty[}(A)(H_{\mathbb{C}}) = H_{\mathbb{C}}$$

It follows that E is dense in $H_{\mathbb{R}}$. Let $(e_i)_{i\in I}$ be an algebraic basis of unit vectors of E and denote by $\mathcal{E}$ the algebra generated by the Gaussians $G(e_i)$ for $i \in I$. $\mathcal{E}$ is w^* dense in Γ and every element in $\mathcal{E}$ is entire for $(\sigma_t)_{t\in\mathbb{R}}$ (because for all $k \in \mathbb{N}_*$, A is bounded and has a bounded inverse on $\chi_{[\frac{1}{k},k]}(A)(H_{\mathbb{C}})$). Denoting by W the Wick product in Γ, we have for all $i \in I$ and all $z \in \mathbb{C}$:

$$\sigma_z(G(e_i)) = W(U_{-z}e_i) = W(A^{-iz}e_i) \tag{25}$$

Since $H_{\mathbb{R}} \subset H$ and for all $n \in \mathbb{N}_*$, $H_{\mathbb{R}} \subset H_n$(isometrically), we have by (1)

$$\text{For all} \quad (i,n) \in I \times \mathbb{N}_*, \qquad \|G_n(e_i)\| = \frac{2}{\sqrt{1-q}} \tag{26}$$

Scholie 6.2 *For all $r \in \mathbb{R}$ and for all $i \in I$ we have*

$$\sup_{n\in\mathbb{N}_*} \|\sigma^n_{ir}(G_n(e_i))\| < +\infty$$

Proof. Fix $i \in I$. By (25):

$$\begin{aligned}
\|\sigma^n_{ir}(G_n(e_i))\| &= \|W(A^r_n e_i)\| = \|a^*_n(A^r_n e_i) + a_n(\mathcal{J}A^r_n e_i)\| \\
&\leqslant C^{\frac{1}{2}}_{|q|}(\|A^r_n e_i\|_{H_n} + \|\mathcal{J}A^r_n e_i\|_{H_n}) \\
&\leqslant C^{\frac{1}{2}}_{|q|}(\|A^r_n e_i\|_{H_n} + \|\Delta^{\frac{1}{2}}_n A^r_n e_i\|_{H_n}) \\
&\leqslant C^{\frac{1}{2}}_{|q|}(\|A^r_n e_i\|_{H_n} + \|A^{r-\frac{1}{2}}_n e_i\|_{H_n})
\end{aligned}$$

Thus it suffices to prove that for all $r \in \mathbb{R}$ we have

$$\sup_{n\in\mathbb{N}_*} \|A^r_n e_i\|_{H_n} < +\infty$$

Let us denote by $\mu_i = \langle E_A(.)e_i, e_i\rangle_{H_{\mathbb{C}}}$ and by $g_r(\lambda) = 2\lambda^{2r+1}/(1+\lambda)$. There exists $k \in \mathbb{N}_*$ such that $e_i \in \chi_{[1/k,k]}(A)(H_{\mathbb{R}})$, thus we have :

$$\|A^r_n e_i\|^2_{H_n} = \langle g_r \circ f_n(A)e_i, e_i\rangle_{H_{\mathbb{C}}} = \int_{[1/k,k]} g_r \circ f_n(\lambda)\mathrm{d}\mu_i(\lambda)$$

It is easily seen that $(g_r \circ f_n)_{n\in\mathbb{N}_*}$ converges uniformly to g_r on $[1/k,k]$. The result follows by:

$$\lim_{n\to+\infty} \|A^r_n e_i\|^2_{H_n} = \lim_{n\to+\infty} \int_{[1/k,k]} g_r \circ f_n(\lambda)\mathrm{d}\mu_i(\lambda) = \int_{[1/k,k]} g_r(\lambda)\mathrm{d}\mu_i(\lambda) = \|A^r e_i\|^2_H.$$

□

6.2 Conclusion

Recall that $\mathcal{E}$ is isomorphic to the complex free $*$-algebra with $|I|$ generators. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}_*$, by (26) we can define a $*$-homomorphism Φ from $\mathcal{E}$ into the von Neumann algebra ultraproduct over $\mathcal{U}$ of the algebras Γ_n by:

$$\begin{aligned} \Phi \,:\, \mathcal{E} &\longrightarrow \mathcal{A} = \prod_{n,\mathcal{U}} \Gamma_n \\ G(e_i) &\longmapsto (G_n(e_i))^{\bullet}_{n\in\mathbb{N}_*} \end{aligned}$$

We will now check the hypothesis of Theorem 4.3.

1. We first check that Φ is state preserving. It suffices to verify it for a product of an even number of Gaussians. Take $(i_1,\dots,i_{2k})\in I^{2k}$, we have by Scholie 6.1:

$$\begin{aligned} \varphi(G(e_{i_1})\dots G(e_{i_{2k}})) &= \sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\dots,k)\\ \mathcal{V}=((s(l),t(l)))_{l=1}^{l=k}}} q^{i(\mathcal{V})}\prod_{l=1}^{l=k}\langle e_{i_{s(l)}}, e_{i_{t(l)}}\rangle_H \\ &= \lim_{n\to+\infty} \sum_{\substack{\mathcal{V}\in\mathcal{P}_2(1,\dots,k)\\ \mathcal{V}=((s(l),t(l)))_{l=1}^{l=k}}} q^{i(\mathcal{V})}\prod_{l=1}^{l=k}\langle e_{i_{s(l)}}, e_{i_{t(l)}}\rangle_{H_n} \\ &= \lim_{n\to+\infty} \varphi_n(G_n(e_{i_1})\dots G_n(e_{i_{2k}})) \end{aligned}$$

   This implies, in particular that Φ is state preserving.

2. Condition 1. of lemma 4.1 is satisfied by Scholie 6.2.

3. It suffices to check that for all $i\in I$ and all $t\in\mathbb{R}$, $(\sigma_t^n(G_n(e_i)))^{\bullet}_{n\in\mathbb{N}_*}\in\overline{\mathrm{Im}\Phi}^{w^*}$. Fix $i\in I$ and $t\in\mathbb{R}$. For all $n\in\mathbb{N}_*$ we have

$$\|A_n^{-it}e_i - A^{-it}e_i\|^2_{H_\mathbb{R}} = \int_{\mathbb{R}_+} |f_n^{-it}(\lambda)-\lambda^{-it}|^2 \mathrm{d}\mu_i(\lambda)$$

   By the Lebesgue dominated convergence Theorem, it follows that

$$\lim_{n\to+\infty}\|A_n^{-it}e_i - A^{-it}e_i\|_{H_\mathbb{R}} = 0.$$

   By (26) we deduce that

$$\lim_{n\to+\infty}\|G_n(A_n^{-it}e_i) - G_n(A^{-it}e_i)\| = 0$$

   Thus we have

$$(\sigma_t^n(G_n(e_i)))^{\bullet}_{n\in\mathbb{N}_*} = (G_n(A_n^{-it}e_i))^{\bullet}_{n\in\mathbb{N}_*} = (G_n(A^{-it}e_i))^{\bullet}_{n\in\mathbb{N}_*}\in\overline{\mathrm{Im}\Phi}^{\|\cdot\|}\subset\overline{\mathrm{Im}\Phi}^{w^*}.$$

By Theorem 4.3, we deduce our main Theorem:

Theorem 6.3 *Let $H_\mathbb{R}$ be a real Hilbert space given with a group of orthogonal transformations $(U_t)_{t\in\mathbb{R}}$. Then for all $q\in(-1,1)$ the q-Araki-Woods algebra $\Gamma_q(H_\mathbb{R},(U_t)_{t\in\mathbb{R}})$ is QWEP.*

Remark. We were unable to prove that the C^*-algebra $C_q^*(H_{\mathbb{R}}, (U_t)_{t\in\mathbb{R}})$ (for $(U_t)_{t\in\mathbb{R}}$ non trivial) is QWEP, even for a finite dimensional Hilbert space $H_{\mathbb{R}}$. The proof of Lemma 3.5 could not be directly adapted to this case. Indeed, in the non-tracial framework, the ultracontractivity of the q-Ornstein-Uhlenbeck semi-group $(\Phi_t)_{t\in\mathbb{R}_+}$ is known when A is bounded and $t > \frac{\ln(\|A\|)}{2}$ but in any cases it fails for $0 < t < \frac{\ln(\|A\|)}{4}$ (see [Hi] Theorem 4.1 and Proposition 4.5).

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